# HYPERBOLIC BEHAVIOUR OF GEODESIC FLOWS ON MANIFOLDS WITH NO FOCAL POINTS

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## DECLARATION

The main results of this thesis appear, in slightly weaker form, in a paper with the same title which has been accepted for publication in Ergodic Theory and Dynamical Systems.

ABSTRACT

We consider a Riemannian manifold $M$ with no focal points such that the universal cover contains a geodesic which does not bound a flat totally geodesically embedded half plane. It is shown that

(i) If the non-wandering set of the geodesic flow is the whole of $SM$, then the closed orbits are dense in $SM$.

(ii) If $M$ is compact then the geodesic flow is ergodic and Bernoulli with respect to the Liouville measure on $SM$.

## INTRODUCTION

Anosov [1] showed in the early 1960's that the geodesic flow on the unit tangent bundle of a compact manifold of negative curvature has the following properties:

(i) There is a dense orbit.

(ii) The closed orbits are dense.

(iii) The flow is ergodic.

This raised the question of whether these properties could be proved under less restrictive conditions. The natural starting point was to seek reasonable conditons under which they hold for a manifold with non-positive curvature. For the moment suppose that $M$ is such a manifold.

In [19] Eberlein and O'Neill introduced the axiom of uniform visibility. If $M$ is compact, it satisfies this axiom if and only if its universal cover contains no totally geodesically embedded flat plane; for the general definition see (8.11). Eberlein showed [5, 16, 17] that if $M$ satisfies uniform visibility and the non-wandering set $\Omega$ of the geodesic flow is the whole of $SM$, then the geodesic flow has a dense orbit and is topologically mixing. Further, any geodesic which does not bound a totally geodesically immersed flat strip is a limit of closed geodesics.

Ballmann [4, 5] subsequently considered the weaker property that there be at least one geodesic in $M$ which does not bound a flat totally geodesically immersed half plane. He showed that all of Eberlein's results carry over to this case.

Pesin [38,39] almost proved that the geodesic flow of a compact uniform visibility manifold is ergodic. What he showed was that if the set $\Lambda$, where the characteristic exponents of the geodesic flow (except in the flow direction) are all non-zero, has positive measure, then it has full measure and the geodesic flow is ergodic and Bernoulli. Ballmann and Brin [6] have shown that Pesin's theorem is still true if uniform visibility is weakened to Ballmann's condition that some geodesic not bound a flat half plane.

In this thesis we will consider a wider class of Riemannian manifolds than those with non-positive curvature — namely, those with no focal points. A manifold has no focal points if for any initially vanishing Jacobi field $Y$, $\|Y(t)\|$ is strictly increasing for $t > 0$. Geometrically this means that the universal cover has two properties. Firstly any two points are joined by a unique geodesic (an equivalent statement is that the manifold has no conjugate points, i.e. any Jacobi field that vanishes twice is identically zero). Secondly every geodesic ball is strictly convex. All of the results described above remain true for manifolds with no focal points. Most of Eberlein's results about uniform visibility were in fact proved for the wider class of manifolds with no conjugate points [14, 15, pp. 508-509]. To extend Ballmann's results requires a new lemma which we prove here and some other changes; see §7A. Pesin's theorem was already proved for manifolds with no focal points. Ballmann and Brin's extension of it can still be proved with essentially the same argument.

The main result of this thesis is that if $M$ is compact, has no focal points, and satisfies Ballmann's condition, then the set $\Lambda$ has positive measure. It follows immediately that the geodesic flow is ergodic and Bernoulli. The same result has been proved independently by Ballmann and Brin. A weaker result appears in [6]. One of the steps in our argument will be to show that the closed orbits of the geodesic flow are dense in $SM$ when $M$ satisfies Ballman's condition and $\Omega = SM$.

Thus, if $M$ has no focal points and satisfies Ballmann's condition, properties (i), (ii) and (iii) above all hold if $M$ is compact, and (i) and (ii) hold if $\Omega = SM$. I do not know whether (iii) still holds if $\Omega = SM$. Nor do I know whether these results extend to manifolds with no conjugate points. This seems likely for surfaces: then the geodesic flow is known to be ergodic and Bernoulli unless its measure entropy is $0$ [39, Theorem 9.4]. In higher dimensions, however, there appear to be grave difficulties. It is not even clear whether horospheres can be constructed without some extra hypothesis.

For a comprehensive survey of results related to the geodesic flow on manifolds with no focal points, see [41].

Since the research in this thesis was done, great progress has been made in a joint work by Ballmann, Brin, Eberlein and Spatzier. They studied a compact manifold $M$ with non-positive curvature. They showed that for such a manifold the closed orbits of the geodesic flow are always dense, and that Ballmann's condition, topological transitivity of the geodesic flow, and ergodicity are equivalent. Furthermore their results suggest that $M$ can only fail to satisfy Ballmann's condition if it is a symmetric space or a product. It seems very likely that their results will extend to manifolds with no focal points.

* * *

Now an outline of the thesis. I aim to give a fairly detailed account of the geometric aspects of geodesic flows. Geometrical methods alone are sufficient to prove topological dynamical properties such as transitivity and density of closed orbits. Even the proof that the set $\Lambda$ has positive measure requires only elementary ideas about characteristic exponents. Of course, the theorem of Pesin stated above requires a very difficult idea (absolute continuity) from ergodic theory.

The first three sections are introductory; they contain no new results. §§1 and 2 contain basic facts about geodesics and Jacobi fields. §3 is an exposition of the theory of Jacobi tensors that has been developed by Eschenburg and O'Sullivan [20, 21, 22, 23], based on results of Green [28] and Eberlein [17]. I have made a change in the definition of a Jacobi tensor, and to avoid confusion the objects studied here will be called Jacobi maps. The change is superficial, but I prefer the form in which the theory is presented here. Green's original idea was to replace the normal Jacobi equation with an analogous matrix equation. Jacobi tensors and Jacobi maps are closely related objects which in appropriate coordinate systems are represented by Green's matrices. The difference is in the choice of coordinate system. See Remark (3.1 iv).

The point of the theory is to simultaneously study all the Jacobi fields belonging to a subspace of the space $J^{\perp}(\gamma)$ of orthogonal Jacobi fields along a geodesic $\gamma$. A Jacobi map is a family of maps $J(t): \dot{\gamma}(0)^{\perp} \to \dot{\gamma}(t)^{\perp}$ ( $\perp$ denotes orthogonal complement) such that $t \to J(t)v$ is a Jacobi field for each fixed $v \in \dot{\gamma}(0)^{\perp}$. Each (non-degenerate) Jacobi map corresponds to a basis for an (n-1)-dimensional subspace of $J^{\perp}(\gamma)$. The subspaces that are most interesting geometrically arise in the following way. Suppose we have a family of hypersurfaces orthogonal to $\gamma$ whose orthogonal curves are unit speed geodesics (e.g. the spheres centered at a point on $\gamma$). Then consider the space of Jacobi fields that can be obtained by varying $\gamma$ through the geodesics orthogonal to this family. The subspaces of this form are precisely the Lagrangian subspaces of $J^{\perp}(\gamma)$ with respect to the natural symplectic structure; the corresponding Jacobi maps are called Lagrange maps. Given a Lagrange map, one can easily determine the second fundamental tensors of the hypersurfaces from which it was obtained; see §3D.

It is possible to calculate with Lagrange maps. In §§3D and 3E we present two comparison theorems of Sturmian type and apply the method of reduction of order to obtain a formula expressing one Lagrange map in terms of another. These results will be applied later to three special Lagrange maps. The first, defined in §3C, describes the initially vanishing Jacobi fields along a geodesic. The other two, introduced in §5, correspond to the stable and unstable Jacobi fields.

§§4, 5 and 6 study the geometry of simply connected manifolds with no focal points. The theory we are interested in was developed for manifolds of non-positive curvature by Eberlein and O'Neill [19]. Their idea was to construct, for an arbitrary, simply connected manifold with non-positive curvature, analogues of the boundary circle and the horocycles of the Poincare disc. One defines geodesics $\gamma$ and $\delta$ in $H$ to be asymptotic if $d(\gamma(t),\delta(t))$ is bounded for $t > 0$. One can show that if $p \in H$ and $\gamma$ is a geodesic in $H$, then there is a unique geodesic $\delta$ through $p$ asymptotic to $\gamma$. Using this one can identify the set $H(\infty)$ of all equivalence classes of asymptotic geodesics with a sphere and construct a natural topology (the cone topology) on $\bar{H} = H \cup H(\infty)$ so that it is a closed disc with $H(\infty)$ as boundary. It can be shown that each family of asymptotic curves has a family of orthogonal hypersurfaces: these are the horospheres.

Results of Eberlein [14], Eschenburg and O'Sullivan [20, 21, 22, 23], Pesin [39], and Goto [25,26] have shown that this theory extends intact to manifolds with no focal points, although some of the results are harder to prove. What is different is that in the non-positive curvature case the length of any Jacobi field is convex, and so is $d(\gamma(t),\delta(t))$ for any two geodesics $\gamma$ and $\delta$. These results are no longer true for manifolds with no focal points [37, p. 325]. It is still true, however, that geodesic balls are strictly convex. It follows from this that if $\gamma$ and $\delta$ are unit speed geodesics with $\gamma(0) = \delta(0)$, then $d(\gamma(t),\delta(t))$ is increasing for $t > 0$, and if $\gamma$ and $\delta$ are asymptotic, then $d(\gamma(t),\delta(t))$ is non-increasing for all $t$. In most situations these are adequate replacements for convexity. What is more difficult to show is that two different geodesics with the same initial point cannot be asymptotic.

We give an exposition of this work; with the two exceptions noted below the results were not proved by the present author. § 4 contains the basic properties of manifolds with no conjugate and no focal points. We show that these are closed conditions on Riemannian metrics. This must be well-known but seems not to have been explicitly formulated before. At the end we give some simple consequences of convexity of geodesic balls in manifolds of no focal points, in particular that the distance between intersecting geodesics increases as one moves away from the intersection point.

§5 introduces the stable Jacobi fields. They are defined as certain limiting solutions of Jacobi's equation. In §6 it will be seen that they are the Jacobi fileds obtained by varying a geodesic through geodesics asymptotic to it, see (6.19). The main result (5.13) is that if different geodesics $\gamma$ and $\delta$ have $\gamma(0) = \delta(0)$ then $d(\gamma(t),\delta(t)) \to \infty$ as $t \to \infty$, and so $\gamma$ and $\delta$ cannot be asymptotic. This follows from the corresponding result for Jacobi fields (5.11) which is proved by calculations using the techniques from §3. The argument is due to Goto; we have slightly sharpened her result.

§6 develops the theory of points at infinity and horospheres for manifolds with no focal points, using the results of §§4 and 5. We emphasize (6.18) and (6.20) which show that horospheres are $C^2$-hypersurfaces and vary continuously with the family of asymptotic geodesics defining them.

The last three sections study manifolds satisfying Ballmann's condition. §7 contains technical results of Ballmann which are needed in §§8 and 9 where the results discussed in the first half of the introduction are proved. For a more detailed survey of the contents of these sections, the reader should consult §7A and the introduction to §§8 and 9.

CONTENTS

## §1. GEODESIC FLOW AND THE UNIT TANGENT BUNDLE

This section summarizes some basic facts about geodesic flows. It is based on [8, Chapter 1], [34, Chapter 3.1], [3] and [35].

### A. THE GEODESIC FLOW

Let $M$ be a Riemannian manifold. Our manifolds will always be complete, connected and smooth ($= C^\infty$). For conventions in differential geometry regarding differential forms and the sign of the curvature tensor, we follow [1]. Let $n$ denote the dimension of $M$, and $d$ the distance function. If $v, w \in T_pM$ are both non-zero, the angle $\measuredangle_p(v,w)$ is the unique number between $0$ and $\pi$ such that $\cos \measuredangle_p(v,w) = \langle v,w\rangle/(\|v\|.\|w\|)$. Define $G:TM \to T^*M$ by

$$G(v) = \langle\cdot,v\rangle.$$

If $\sigma(t)$ is a curve (in $M$ or $TM$), $\dot\sigma(t)$ will denote its tangent vector. We will use $D_\sigma$, $\frac{D}{dt}$ or a dash ' to denote covariant differentiation along a curve $\sigma(t)$ in $M$.

A *geodesic* of $M$ is a parametrized curve $\gamma$ in $M$ such that

$$D_\gamma \dot\gamma = 0. \qquad \text{(G)}$$

A geodesic is the path of a particle moving in $M$ without acceleration. Clearly $\|\dot\gamma(t)\|$ is constant, and $\gamma$ is parametrized by arclength when $\|\dot\gamma(t)\| = 1$. For any $v \in TM$ there is a unique geodesic $\gamma_v$ with $\dot{\gamma}_v(0) = v$. The Hopf-Rinow theorem tells us that, since M is complete, $\gamma_v(t)$ is defined for all t. Thus there is a flow $\Phi_t$ defined on the tangent bundle TM by

$$\Phi_t(v) = \dot{\gamma}_v(t).$$

The flow $\Phi_t$ leaves invariant all of the sets $\{v \in TM: \|v\| = c\}$. Its action is the same on each of them, apart from a uniform change of speed. Thus it is natural to study $\Phi_t$ acting on the unit tangent bundle $SM = \{v \in TM: \|v\| = 1\}$, rather than all of TM.

1.1 DEFINITION

The *geodesic flow* of M is the restriction to SM of the flow $\Phi_t$.

This section considers some basic properties of $\Phi_t$. The main observation is that $\Phi_t$ is generated by a Hamiltonian vector field on TM, and so leaves invariant a certain 2-form on TM. This leads to natural invariant measures for $\Phi_t$ on TM and SM.

For later use we will now make some definitions and conventions. If $p \in M$, the *exponential map* $\exp_p: T_pM \to M$ is defined by $\exp_p(v) = \gamma_v(1)$. The *geodesic sphere* $S(p,r)$ and the *geodesic ball* $B(p,r)$ are the images under $\exp_p$ of $\{v \in T_pM: \|v\| = r\}$ and $\{v \in T_pM: \|v\| < r\}$ respectively. There is a fundamental result, known as *Gauss' Lemma*.

1.2 <u>THEOREM</u> [35, Lemma 10.5]

For each $p \in M$ there is $r > 0$ such that $\exp_p:\{v \in TM: \|v\| < r\} \to B(p,r)$ is a diffeomorphism. The geodesics through p are orthogonal to the geodesic spheres $S(p,s)$ for $0 < s < r$.

To avoid inconvenience we assume henceforth that geodesics are non-constant: *if* $\gamma$ *is a geodesic, then* $\dot\gamma(t) \neq 0$.

## B. CLASSICAL MECHANICS

At this point let us recall some basic results of classical mechanics. Later we shall interpret them in the case of the geodesic flow. Consider a particle with unit mass moving in M under the influence of the potential $U:M \to \mathbb{R}$. The trajectory $\gamma(t)$ of the particle satisfies Newton's second law of motion

$$D_\gamma \dot\gamma(t) = -G^{-1}(dU(\gamma(t))). \qquad \text{(N)}$$

The geodesic equation (G) is the special case when $U = 0$.

Define the *Lagrangian* $L:TM \to \mathbb{R}$ by

$$L(v) = \tfrac{1}{2} \langle v,v\rangle - U(\pi v),$$

so $L(v)$ is the difference between the potential and kinetic energies of the particle. If $\sigma:[a,b] \to M$ is a piecewise smooth curve, define the *action* of L along $\sigma$,

$$A(\sigma) = \int_a^b L(\dot\sigma(t))dt.$$

The methods of the calculus of variations show that Newton's law (N) is the Euler-Lagrange equation for A. This is the *principle of least action:* $\gamma:[a,b] \to M$ is a trajectory of the particle if and only if it is an extremal curve for A (with respect to variations fixing the endpoints).

Now turn to the Hamiltonian viewpoint. Newton's equation (N) is a second order differential equation on M, so it defines a vector field $\Xi$ on TM. To describe $\Xi$ one introduces the *Hamiltonian* function $H:TM \to \mathbb{R}$ and a *symplectic structure* (closed, non-degenerate 2-form) $\omega$ on TM. The Hamiltonian is the total energy of the particle:

$$H(v) = \tfrac{1}{2}\langle v,v\rangle + U(\pi v).$$

To define $\omega$ recall that the cotangent bundle $T^*M$ has a canonical symplectic structure

$$\Omega = -d\Theta,$$

where $\Theta$ is the canonical 1-form on $T^*M$. To define $\Theta$, think of the commutative diagram

$$\begin{array}{ccc} TT^*M & \xrightarrow{T\pi_{T^*M}} & TM \\ {\scriptstyle \pi_{TT^*M}}\downarrow & & \downarrow{\scriptstyle \pi = \pi_{TM}} \\ T^*M & \xrightarrow{\pi_{T^*M}} & M \end{array}$$

If $\lambda \in TT^*M$, $\Theta(\lambda) = (\pi_{TT^*M}\lambda)(T\pi_{T^*M}\lambda)$.

Now define

$$\theta = G^*\Theta$$

and $\omega = -d\theta = G^*\Omega.$

Finally the vector field $\Xi$ is defined by *Hamilton's equation*

$$i_\Xi \omega = dH, \qquad \text{(H)}$$

where

$$i_\Xi \omega = \omega(\Xi,\cdot).$$

It is a theorem that the integral curves of $\Xi$ are the curves formed by the tangents to the extremal curves of A, and so give the velocity of the particle as it moves under the potential U.

Both the Hamiltonian and the symplectic structure are invariant under the flow defined by $\Xi$.

1.3 <u>PROPOSITION</u>

(i) $L_\Xi H = 0$. (Conservation of energy)

(ii) $L_\Xi \omega = 0$.

<u>Proof</u>

(i) By (H), $L_\Xi H = \omega(\Xi,\Xi) = 0$.

(ii) By [1, Theorem 2.4.13], $L_\Xi \omega = i_\Xi d\omega + di_\Xi\omega$

$$= -i_\Xi dd\theta + ddH$$

$$= 0. \qquad \square$$

In particular the measure defined on TM by $\omega^n$ is $\Xi$-invariant.

Remark

Hamilton's equation is usually developed in a slightly different way that does not require M to have a Riemannian metric. Also the Lagrangian $L:TM \to \mathbb{R}$ does not have to be of the special form $\frac{1}{2}\langle v,v\rangle - U(\pi v)$ that we had above.

For $v \in T_pM$, define the fibre derivative of L at v,

$$FL(v) = d(L|T_pM)(v) \in T_p^*M.$$

The map FL is called the *Legendre transform.* Define a function H and 2-form $\omega$ on TM by

$$H(v) = (FL(v))(v) - L(v)$$

and

$$\omega = (FL)^*\Omega.$$

Then an extremal curve, $\gamma$, of the action integral A satisfies

$$i_{\ddot{\gamma}}\omega = dH.$$

If FL is a local diffeomorphism (e.g. if L is convex on each fibre $T_pM$ as in our situation), $\omega$ will be non-degenerate and one can define the vector field $\Xi$ as before. For a full account see Chapter 1 of [8].

When $L(v) = \frac{1}{2}\langle v,v\rangle - U(\pi v)$, this approach is the same as our earlier treatment because

$$FL(v) = G(v).$$

## C. GEODESICS AS EXTREMAL CURVES

Let us return to geodesics and the flow $\Phi_t$ on TM. This is the special case of the theory in paragraph B when the potential U vanishes. Both L(v) and H(v) are equal to

$$e(v) = \tfrac{1}{2} \langle v,v \rangle .$$

We consider first the properties of geodesics as extremal curves.

If $\sigma:[a,b] \to M$ is a piecewise smooth curve, define its *energy*

$$E(\sigma) = \int_a^b e(\dot{\sigma}(t))dt$$

(this gives the potential energy stored in a uniform spring stretched to lie along $\sigma$) and its *arclength*

$$\ell(\sigma) = \int_a^b \|\dot{\sigma}(t)\| \, dt.$$

The energy integral is just the action integral with $U = 0$. Thus geodesics are the extremal curves for the energy integral by the least action principle (for a direct proof see [35, Corollary 12.3]). Geodesics are also extremal curves for $\ell$. In fact the extremal curves for $\ell$ are precisely all possible reparametrizations of geodesics; see the discussion after Theorem 12 of Chapter 9 in [42].

In particular, suppose $\gamma:[a,b] \to M$ is parametrized by arclength and there is no shorter curve joining $\gamma(a)$ to $\gamma(b)$.

Then $\ell(\gamma) = d(\gamma(a), \gamma(b))$, and $\gamma$ must be a geodesic. A geodesic which is the shortest path between its endpoints is called *minimal*. Short enough geodesic segments are always minimal. The proof is based on Gauss' Lemma (1.2); see [35, Theorem 10.4]. Since M is complete, the Hopf-Rinow theorem guarantees that any two points of M are joined by a minimal geodesic.

## D. THE VECTOR FIELD WHICH GENERATES THE GEODESIC FLOW

It follows from the general theory outlined in paragraph B that the flow $\phi_t$ on TM is generated by the vector field $\Xi$ defined by Hamilton's equation with Hamiltonian function $H(v) = e(v) = \frac{1}{2}\langle v,v\rangle$. We now verify this directly, by giving an explicit description of $\Xi$.

Let $K:TTM \to TM$ be the connector map for the Levi-Civita connection. If $\xi \in T_vTM$ and W is any curve in TM with $\dot{W}(0) = \xi$, then $K\xi = D_{\pi \circ W}\, W(0) \in T_{\pi v}M$. It can be verified that $\xi = 0$ if and only if $T\pi\xi = 0 = K\xi$. It follows that for each $v \in TM$, the map $i_v:T_vTM \to T_{\pi v}M \oplus T_{\pi v}M$, $i_v(\xi) = (T\pi\xi, K\xi)$ is a linear isomorphism. The vectors $T\pi\xi$ and $K\xi$ are called the *horizontal* and *vertical components* of $\xi$ respectively; $\ker T\pi$ and $\ker K$ are called the *vertical* and *horizontal subspaces* of $T_vTM$. For a detailed account see [29, §2.4].

If $\gamma(t)$ is a geodesic, the horizontal component of the acceleration vector $\ddot{\gamma}(t)$ is $\dot{\gamma}(t)$ (this much is true for any curve) and the geodesic equation (G) tells us that the vertical component of $\ddot{\gamma}(t)$ is 0. In other words, $\phi_t$ is the flow of the vector field $v \to i_v^{-1}(v,0)$. We now verify that this is the same as the vector field $\Xi$ defined by Hamilton's equation.

1.4 PROPOSITION

If $\xi, \eta \in T_vTM$, then

(i) $\theta(\xi) = \langle T\pi\xi, v\rangle$;

(ii) $\omega(\xi,\eta) = \langle T\pi\xi, K\eta\rangle - \langle T\pi\eta, K\xi\rangle$;

(iii) $de(\xi) = \langle v, K\xi\rangle$.

Proof

(i) $\theta(\xi) = \Theta(TG\xi)$

$= (\pi_{TT^*M}TG\xi)(T\pi_{T^*M}TG\xi)$

$= G(v)(T\pi_{T^*M}TG\xi)$

$= G(v)T\pi\xi$ (since $\pi_{T^*M} \circ G = \pi$)

$= \langle T\pi\xi, v\rangle$.

(ii) Choose vector fields on TTM, $\tilde{\xi}$ and $\tilde{\eta}$, with $\tilde{\xi}(v) = \xi$ and $\tilde{\eta}(v) = \eta$, that are *vertically constant* (i.e. their vertical and horizontal components are constant on each fibre of TTM). Define the vector fields on M, $X = T\pi\tilde{\xi}$ and $Y = T\pi\tilde{\eta}$.

Now $\omega(\xi,\eta) = -d\theta(\xi,\eta)$

$= -\{\tilde{\xi}\theta(\tilde{\eta}) - \tilde{\eta}\theta(\tilde{\xi}) - \theta([\tilde{\xi},\tilde{\eta}])\}(v)$.

Let $V(t)$ be the integral curve of $\tilde{\xi}$ starting at v. Then by (i)

$$\tilde{\xi}\theta(\tilde{\eta}) = d/dt\ \langle T\pi\tilde{\eta}(V(t)),\ V(t)\rangle|_{t=0}$$

$$= d/dt\ \langle Y(\pi\circ V(t)),\ V(t)\rangle|_{t=0}$$

$$= \langle D_{\pi\circ V}\ Y(0), V(0)\rangle + \langle Y(\pi\circ V(0))\ ,\ D_{\pi\circ V}\ V(0)\rangle$$

$$= \langle \nabla_X Y(\pi v), v\rangle + \langle T\pi\eta,\ K\xi\rangle.$$

Similarly,

$$\tilde{\eta}(\theta(\tilde{\xi})) = \langle \nabla_Y X(\pi v), v\rangle + \langle T\pi\xi, K\eta\rangle.$$

Since $\tilde{\xi}$ and $\tilde{\eta}$ are $T\pi$-related to X and Y respectively,

$$\theta([\tilde{\xi},\tilde{\eta}]) = \langle T\pi[\tilde{\xi},\tilde{\eta}](v), v\rangle$$

$$= \langle [X,Y](\pi v), v\rangle.$$

Hence

$$\omega(\xi,\eta) = \langle T\pi\xi, K\eta\rangle - \langle T\pi\eta, K\xi\rangle$$

$$+ \langle(\nabla_Y X - \nabla_X Y + [X,Y])(\pi v), v\rangle,$$

which completes the proof, since $[X,Y] = \nabla_X Y - \nabla_Y X$.

(iii) Choose $\tilde{\xi}$ and V as in (ii). Then

$$de(\tilde{\xi}) = d/dt\ \tfrac{1}{2}\ \langle V(t),\ V(t)\rangle|_{t=0}$$

$$= \langle V(0),\ D_{\pi\circ V}\ V(0)\rangle$$

$$= \langle v, K\xi\rangle. \qquad \square$$

Note that it is clear from (ii) that $\omega$ is non-degenerate (i.e. $\omega(\xi,\cdot) = 0$ if and only if $\xi = 0$) and so $\omega$ is a symplectic form.

1.5 COROLLARY

$$\Xi(v) = i_v^{-1}(v,0).$$

Proof.

Substituting the results of (1.4) into Hamilton's equation gives

$$\langle T\pi\Xi(v), K\xi\rangle - \langle T\pi\xi, K\Xi(v)\rangle = \langle v, K\xi\rangle$$

for *every* $\xi \in T_vTM$. □

Thus $\Xi$ defines the geodesic flow $\Phi_t$ on TM. It follows from Proposition 1.2 that *the symplectic structure* $\omega$ *is* $\Phi_t$*-invariant.* This will be important in the study of the derivative of $\Phi_t$ in §2. Also it leads to natural $\Phi_t$-invariant measures on TM and SM as we now see.

## E. THE SASAKI METRIC AND INVARIANT MEASURES

There is a natural Riemannian metric on TM known as the *Sasaki metric*

$$\langle \xi,\eta\rangle = \langle T\pi\xi, T\pi\eta\rangle + \langle K\xi, K\eta\rangle$$

for $\xi, \eta \in T_vTM$. This definiton makes $i_v$ an isometry. A basic result is that *the volume forms* $\mathrm{VOL}_{TM}$ *and* $\mathrm{VOL}_{SM}$ *defined* *by the Sasaki metric on* TM *and* SM *respectively are both* $\Phi_t$-*invariant.* To prove this we shall express them in terms of the invariant symplectic form $\omega$.

1.6 LEMMA

$$\mathrm{VOL}_{TM} = \pm\ 1/n!\ \omega^n.$$

Proof

Let $v \in T_pM$. Choose an orthonormal basis $u_1,\dots,u_n$ for $T_pM$. Write $\xi_i = i_v^{-1}(u_i,0)$ and $\eta_i = i_v^{-1}(0,u_i)$. Then $\{\xi_1,\dots,\xi_n, \eta_1,\dots,\eta_n\}$ is an orthonormal basis for $T_vTM$. Let $\{\lambda^{(1)},\dots,\lambda^{(n)}, \mu^{(1)},\dots,\mu^{(n)}\}$ be the dual basis for $T_v^*TM$. A computation gives $\omega = \sum_{i=1}^{n} \lambda^{(i)} \wedge \mu^{(i)}$. Hence

$$1/n!\ \omega^n = \lambda^{(1)} \wedge \mu^{(1)} \wedge \lambda^{(2)} \wedge \mu^{(2)} \wedge\dots\wedge \lambda^{(n)} \wedge \mu^{(n)}.$$

But this is $\mathrm{VOL}_{TM}$, up to sign. □

It follows immediately that $\mathrm{VOL}_{TM}$ is $\Phi_t$-invariant.

1.7 LEMMA

$$\mathrm{VOL}_{SM} = \mp\ \left(\frac{1}{(n-1)!}\right)\ \theta \wedge \omega^{n-1}.$$

Proof

For $v \in SM$, let $N(v) = i_v^{-1}(0,v)$. Note that $N(v)$ has unit length and is orthogonal to SM in the Sasaki metric.

Hence

$$\begin{aligned} VOL_{SM} &= i_N\, VOL_{TM} \\ &= \pm \frac{1}{n!}\, i_N \omega^n \text{ by (1.6)} \\ &= \pm \frac{1}{(n-1)!}\, (i_N\omega) \wedge \omega^{n-1}. \end{aligned}$$

By (ii) and (i) of (1.3) we have

$$\omega(N(v),\eta) = \langle 0, K\eta\rangle - \langle T\pi\eta, v\rangle = -\theta(\eta)$$

for $\eta \in T_v TM$. Thus $i_N\omega = -\theta$, and so

$$VOL_{SM} = \mp \frac{1}{(n-1)!}\, \theta \wedge \omega^{n-1}. \qquad \square$$

1.8 LEMMA

$$L_{\Xi}\theta = de.$$

Proof

Note that $i_{\Xi}\theta(v) = \langle v,v\rangle = 2e(v)$.

Hence

$$\begin{aligned} L_{\Xi}\theta &= i_{\Xi}\, d\theta + d i_{\Xi}\theta \\ &= -i_{\Xi}\omega + 2\, de \\ &= -de + 2\, de. \qquad \square \end{aligned}$$

Since e is constant on SM, it follows that $\theta$ is $\Phi_t$-invariant on SM. It now follows from (1.7) that $\mathrm{VOL}_{SM}$ is $\Phi_t$-invariant. See also (2.4ii).

We will use $\mu$ to denote the invariant measure defined on SM by $\mathrm{VOL}_{SM}$. We call this measure the *Liouville measure.*

## F. THE UNIVERSAL COVER

Usually H will denote the Riemannian universal cover of M (sometimes M = H). The geodesics of H are the lifts of geodesics in M. Many other objects, for example the Jacobi fields and Jacobi maps considered in §§2 and 3 also lift naturally to H. This presents no difficulty and will not be discussed in detail. Normally the same symbols will be used for corresponding objects in M and H.

## G. THE SECOND FUNDAMENTAL TENSOR OF A HYPERSURFACE

We will use the definition of [29, p. 104]. Suppose S is a $C^2$ hypersurface embedded in M and $\nu$ is a unit normal to S at p. The *second fundamental tensor of* S *relative to* $\nu$ is the symmetric map $II:T_pS \to T_pS$ defined as follows. Let N be the field of unit normals on S with $N(p) = \nu$, and let $\tilde{N}$ be any $C^1$ extension of N to M. If $v \in T_pS$,

$$II(v) = P(\nabla_v\tilde{N}),$$

where $P:T_pM \to T_pS$ is orthogonal projection. With this definition, the more positive the eigenvalues of II are, the more S bends away from $\nu$.

## §2. JACOBI'S EQUATION AND THE DERIVATIVE OF THE GEODESIC FLOW

Jacobi's equation is the differential equation satisfied by the vector fields along a geodesic obtained by variations through geodesics. Its solutions, the Jacobi fields, determine the derivative of the geodesic flow. References are [35, §14] and [29, §4.2].

### A. JACOBI FIELDS

Let $\gamma$ be a geodesic in a manifold $M$. A $C^k$ $(1 < k < \infty)$ *variation of* $\gamma$ *through geodesics* is a $C^k$ map $\alpha:(-\varepsilon,\varepsilon) \times \mathbb{R} \to M$ such that $\alpha(s,t)$ is a geodesic for each fixed $s$ and $\alpha(0,t) \equiv \gamma(t)$.

2.1 PROPOSITION [35, §14]

If $\alpha$ is a $C^\infty$ variation of $\gamma$ through geodesics, then the vector field $Y(t) = \frac{\partial\alpha}{\partial s}(0,t)$ satisfies Jacobi's equation

$$Y''(t) + R(Y(t), \dot{\gamma}(t))\dot{\gamma}(t) = 0. \qquad (J)$$

Conversely, every solution of (J) is of the form $\frac{\partial\alpha}{\partial s}(0,t)$ for some $C^\infty$ variation $\alpha(s,t)$ of $\gamma$ through geodesics.

A solution of (J) is called a *Jacobi field* along $\gamma$. A Jacobi field $Y(t)$ is uniquely determined by $Y(0)$ and $Y'(0)$. The set of all Jacobi fields along $\gamma$ is a 2n-dimensional vector space, which we denote by $J(\gamma)$.

2.2 <u>REMARKS</u>

(i) If $\gamma$ is a geodesic, so is the curve $\bar{\gamma}(t) = \gamma(-t)$. The geodesics $\gamma$ and $\bar{\gamma}$ have the same Jacobi fields. This follows because the geodesic and Jacobi equations contain only even order derivatives.

(ii) It is clear from the geodesic equation, $\dot{\gamma}' = 0$, that $\dot{\gamma}(t)$ and $t\dot{\gamma}(t)$ are both Jacobi fields along $\gamma$.

(iii) Jacobi's equation is easy to solve if M has constant curvature K. Assume for simplicity that $\gamma$ has unit speed. Let $u_1,\ldots,u_{n-1}$ be parallel vector fields along $\gamma$ such that $\dot{\gamma}(t)$, $u_1(t),\ldots,u_{n-1}(t)$ are orthonormal. Define

$$\begin{array}{llll} C_K(t) = \cos\sqrt{K}\,t, & S_K(t) = \sin\sqrt{K}\,t & \text{if } K > 0 \\ \phantom{C_K(t)} = 1 & \phantom{S_K(t)} = t & \text{if } K = 0 \\ \phantom{C_K(t)} = \cosh\sqrt{-K}\,t & \phantom{S_K(t)} = \sinh\sqrt{-K}\,t & \text{if } K < 0. \end{array}$$

Then $\dot{\gamma}(t)$, $t\dot{\gamma}(t)$ and $C_K(t)u_i(t)$, $S_K(t)u_i(t)$ $(1 \leq i \leq n-1)$ are 2n independent Jacobi fields.

2.3 <u>PROPOSITION</u>

Suppose $\gamma$ is a geodesic in M. For each $\xi \in T_{\dot{\gamma}(0)}TM$ there is a unique Jacobi field $Y_\xi$ along $\gamma$ such that

$$T\Phi_t(\xi) = i^{-1}_{\dot{\gamma}(t)}(Y_\xi(t), Y'_\xi(t)).$$

The map $\xi \to Y_\xi$ is a linear isomorphism between $T_{\dot{\gamma}(0)}TM$ and $\mathcal{J}(\gamma)$.

Proof

Uniqueness of $Y_\xi$ is clear. We prove existence. Choose a $C^\infty$ curve $W(s)$ in TM with $W(0) = \dot{\gamma}(0)$ and $\dot{W}(0) = \xi$. Define the $C^\infty$ variation of $\gamma$ through geodesics

$$\alpha(s,t) = \gamma_{W(s)}(t).$$

Then the Jacobi field $Y(t) = \frac{\partial\alpha}{\partial s}(0,t)$ has the desired properties. For $\frac{\partial\alpha}{\partial t}(s,t) \equiv \Phi_t(W(s))$ and so $T\Phi_t(\xi)$ is the derivative at $s = 0$ of the curve in $TM, s \to \frac{\partial\alpha}{\partial t}(s,t)$. Thus

$$T\pi \circ T\Phi_t(\xi) = \frac{\partial\alpha}{\partial s}(0,t) = Y(t)$$

and

$$K \circ T\Phi_t(\xi) = \frac{D}{ds}\frac{\partial\alpha}{\partial t}(0,t) = \frac{D}{dt}\frac{\partial\alpha}{\partial s}(0,t) = Y'(t).$$

Finally, it is clear that $\xi \to Y$ is injective and linear; it is an isomorphism because $T_{\dot{\gamma}(0)}TM$ and $J(\gamma)$ both have dimension $2n$. □

2.4 EXAMPLES

(i) If $v \neq 0$, and $\Xi(v) = i_v^{-1}(v,0)$ as in §1, then $Y_{\Xi(v)} = \dot{\gamma}_v$.

(ii) Suppose $v \in SM$. If $N(v) = i_v^{-1}(0,v)$ as in (1.7), then $Y_{N(v)} = t\dot{\gamma}_v$ and $Y'_{N(v)} = \dot{\gamma}_v$. It follows that in the Sasaki metric

$$\langle T\Phi_t(N(v)), N(\Phi_t(v))\rangle = \langle\dot{\gamma}_v(t), \dot{\gamma}_v(t)\rangle = 1 \qquad (*)$$

for all t. This leads to an alternative proof of the $\Phi_t$- invariance of $VOL_{SM}$ to that in (1.7) and (1.8). Since $VOL_{SM} = i_N\, VOL_{TM}$ and $VOL_{TM}$ is $\Phi_t$-invariant by (1.6), all we need to show is that the component of $T\Phi_t(N(v))$ orthogonal to SM has constant length. Since N is a unit vector field orthogonal to SM, this is immediate from (*).

We now make two applications of (2.3). Firstly we observe that Jacobi fields can be constructed from variations through geodesics that are only $C^1$; this will be needed later.

2.5 <u>LEMMA</u>

Suppose $\gamma$ is a geodesic in M and $V:(-\varepsilon,\varepsilon) \to TM$ is a $C^1$ curve with $V(0) = \dot{\gamma}(0)$. Let

$$\alpha(s,t) = \gamma_{V(s)}(t).$$

Then $\frac{\partial\alpha}{\partial s}(0,t)$ is a Jacobi field along $\gamma$. Also $\frac{D}{ds}\frac{\partial\alpha}{\partial t}(0,t)$ and $\frac{D}{dt}\frac{\partial\alpha}{\partial s}(0,t)$ are defined and equal for all t.

<u>Proof</u>

Clearly $\alpha(s,t) = \pi\circ\Phi_t(V(s))$. Hence $\frac{\partial\alpha}{\partial s}(0,t) = T\pi\circ T\Phi_t(\dot{V}(0))$, which is a Jacobi field by (2.3).

It follows that $K\circ T\Phi_t(\dot{V}(0)) = \frac{D}{dt}\frac{\partial\alpha}{\partial s}(0,t)$. On the other hand, $V_t = \Phi_t\circ V$ is a $C^1$ curve in TM and it is clear that $\frac{D}{ds}\frac{\partial\alpha}{\partial t}(0,t) = K(\dot{V}_t(0)) = K\circ T\Phi_t(\dot{V}(0))$. □

<u>Remark</u>

It is not difficult to show that any $C^1$ variation of $\gamma$

through geodesics is of the form considered in (2.5). See (3.4).

Secondly, we express in terms of Jacobi fields the $T\Phi_t$-invariance of the symplectic form $\omega$ on $TM$.

2.6 <u>LEMMA</u>

Let $Y$ and $Z$ be Jacobi fields along a geodesic $\gamma$. Then

(i) $\langle Y(t), Z'(t)\rangle - \langle Z(t), Y'(t)\rangle$ is constant;

(ii) $\langle Y'(t), \dot\gamma(t)\rangle$ is constant.

<u>Proof</u>

(i) Choose $\eta, \zeta \in T_{\dot\gamma(0)}TM$ with $Y_\eta = Y$ and $Y_\zeta = Z$. By (2.3) and (1.4ii)

$$\langle Y(t), Z'(t)\rangle - \langle Z(t), Y'(t)\rangle = \omega(T\Phi_t\eta, T\Phi_t\zeta),$$

which is constant since we saw at the end of §1D that $\omega$ is $T\Phi_t$-invariant.

(ii) By (2.2i), $\dot\gamma$ is a Jacobi field. Take $Z = \dot\gamma$ in (i). □

We shall write $w(Y,Z)$ for the constant in (i) above. Observe that $w$ is a symplectic form on $\mathcal{J}(\gamma)$.

## B. ORTHOGONAL AND TANGENTIAL JACOBI FIELDS

Each Jacobi field splits uniquely into a linear part tangential to the geodesic and a part orthogonal to it.

### 2.7 LEMMA

Let $Y$ be a Jacobi field along the geodesic $\gamma$. Let $Y_\perp(t)$ and $Y_{||}(t)$ be the components of $Y(t)$ orthogonal and parallel to $\dot\gamma(t)$ respectively.

(i) $Y_\perp$ and $Y_{||}$ are both Jacobi fields along $\gamma$.

(ii) $Y_{||}(t) = (a + bt)\dot\gamma(t)$ for some constants a and b.

(iii) $Y_\perp'(t)$ is orthogonal and $Y_{||}'(t)$ is parallel to $\dot\gamma(t)$ for all t.

Proof

Since Jacobi's equation is linear, $Y_\perp$ is a Jacobi field if $Y_{||}$ is. Since $\dot\gamma' = 0$, $\langle Y,\dot\gamma\rangle'(t) = \langle Y'(t), \dot\gamma(t)\rangle$, which is constant by (2.6ii). Hence $Y_{||}(t) = (a + bt)\dot\gamma(t)$ for some constants a and b. We saw in (2.2ii) that this is a Jacobi field. This proves (i) and (ii); (iii) follows easily from $\dot\gamma' = 0$. □

Call a Jacobi field *orthogonal* if $Y(t) \perp \dot\gamma(t)$ for all t and *tangential* if $Y(t)$ is parallel to $\dot\gamma(t)$ for all t. Usually we shall study only orthogonal Jacobi fields, since it is clear from (ii) above that tangential Jacobi fields are the same on any geodesic.

2.8 LEMMA

The following properties of a Jacobi field $Y$ along $\gamma$ are equivalent.

(i) $Y$ is orthogonal.

(ii) $Y(t) \perp \dot{\gamma}(t)$ and $Y'(t) \perp \dot{\gamma}(t)$ for all $t$.

(iii) $Y(t_o) \perp \dot{\gamma}(t_o)$ and $Y'(t_o) \perp \dot{\gamma}(t_o)$ for some $t_o$.

Proof

(i) ⇒ (ii). By (2.7iii). (ii) ⇒ (iii). Trivial. (iii) ⇒ (i). Clearly $Y_{||}(t_o) = 0 = (Y_{||})'(t_o)$, and so $Y_{||} = 0$. □

The analogous lemma holds for tangential Jacobi fields.

We now introduce some terminology that will be used later. If $v \neq 0$, let $v^{||}$ be the subspace of $T_{\pi v}M$ spanned by $v$ and $v^{\perp}$ its orthogonal complement. Define

$$T_v^{\perp}TM = i_v^{-1}(v^{\perp} \oplus v^{\perp})$$

and

$$T_v^{||}TM = i_v^{-1}(v^{||} \oplus v^{||}).$$

These spaces are orthogonal complements in $T_vTM$. If $\gamma$ is a geodesic, let $J^{\perp}(\gamma)$ and $J^{||}(\gamma)$ denote the spaces of orthogonal and tangential Jacobi fields along $\gamma$. We see from (2.8) that if $\xi \in T_{\dot{\gamma}(0)}TM$, then $Y_{\xi} \in J^{\perp}(\gamma)$ if and only if $\xi \in T^{\perp}_{\dot{\gamma}(0)}TM$. Also $T^{\perp}TM$ is $T\Phi_t$-invariant. Analogous properties hold for $T^{||}TM$.

Clearly the vector spaces $T^{\perp}_{\dot\gamma(0)}TM$ and $J^{\perp}(\gamma)$ are $2(n-1)$-dimensional, and the vector spaces $T^{||}_{\dot\gamma(0)}TM$ and $J^{||}(\gamma)$ are 2-dimensional. It is clear that the 2-forms $\omega$ and w are non-degenerate on $T^{\perp}_{\dot\gamma(0)}TM$ and $J^{\perp}(\gamma)$ respectively, making these symplectic vector spaces.

## C. THE GEODESIC FLOW ON SM

Since we will eventually study the geodesic flow on SM, we reformulate some of the above remarks to apply to SM rather than TM.

If $v \in SM$, $v^{\perp}$ is the tangent space at v to the unit sphere in $T_{\pi v}M$, translated to the origin. It follows that $K: T_v SM \to v^{\perp}$ and

$$i_v : T_v SM \to T_{\pi v} M \oplus v^{\perp}$$

is a linear isomorphism. Define

$$T^{\perp}_v SM = i_v^{-1}(v^{\perp} \oplus v^{\perp})$$

and

$$T^{||}_v SM = i_v^{-1}(v^{||} \oplus \{0\}).$$

These spaces are orthogonal complements in $T_v SM$.

Clearly $T^{\perp}_v SM = T^{\perp}_v TM$ and $T^{||} SM$ is the subbundle of TSM spanned by the vector field $\Xi$ which generates the geodesic flow.

Obviously $\xi \in T_v SM$ if and only if $Y_{\xi}$ can be constructed by varying $\gamma_v$ through unit speed geodesics. Since $i_v(T_v SM) = T_{\pi v}M \oplus v^{\perp}$, we see from (2.3) that $\xi \in T_v SM$ if and only if $Y'_{\xi}(0) \perp v$.

The next lemma will be used later to show that certain Jacobi fields are orthogonal.

2.9 LEMMA

Suppose $\gamma$ is a unit speed geodesic and $Y$ is a Jacobi field along $\gamma$ obtained by varying $\gamma$ through unit speed geodesics. Then $Y$ is orthogonal if and only if $Y(0) \perp \dot{\gamma}(0)$.

Proof

By (2.8), $Y$ is orthogonal if and only if $Y(0) \perp \dot{\gamma}(0)$ and $Y'(0) \perp \dot{\gamma}(0)$. We see from the above that $Y = Y_\xi$ for some $\xi \in T_{\dot{\gamma}(0)}SM$ and so $Y'(0) \perp \dot{\gamma}(0)$. $\square$

Remark

When we restrict from $TM$ to $SM$ we lose one dimension of tangential Jacobi fields, namely those of the form $bt\dot{\gamma}(t)$, where $b$ is a constant. They arise from variations $\alpha(s,t) = \gamma(t + sbt)$ in which $\gamma$ is reparametrized at different speeds.

D. A CONVENTION ABOUT GEODESICS

From now on we will use only unit speed geodesics. The word *geodesic* will be understood to mean *unit speed geodesic*.

## §3. JACOBI MAPS

This section continues the study of Jacobi fields. The method used goes back to the 1958 paper of Green [28] where he rewrote Jacobi's equation in matrix form. The results we present are due originally to Green [28] and Eberlein [17]. Later Eschenburg [20, 21] introduced Jacobi tensors, which are a coordinate free formulation of Green's matrix technique. They have been applied by Eschenburg and O'Sullivan [22, 23]. The Jacobi maps that we will introduce are almost identical to Eschenburg's Jacobi tensors; see (3.1 iv).

The idea behind all these techniques is to study spaces of Jacobi fields, rather than individual Jacobi fields. We will consider orthogonal Jacobi fields; a Jacobi map represents a subspace of $J^{\perp}(\gamma)$. Recall that $J^{\perp}(\gamma)$ is a $2(n-1)$-dimensional symplectic vector space. The subspaces that will be most interesting are the Lagrangian subspaces, the $(n-1)$-dimensional subspaces on which the symplectic form $w$ vanishes.

This section contains the basic facts about Jacobi maps that will be needed later. Particularly important are the comparison theorems in §3D and the reduction of order formula in §3E.

### A. $\gamma$-MAPS

First a summary of the formalism that will be used. Let $\gamma$ be a geodesic and let $N$ be the normal bundle of $\gamma$. Thus $N(t) = \dot{\gamma}(t)^{\perp}$. Let Hom $(N,N)$ be the bundle over $\mathbb{R}^2$ with fibre

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Hom $(N(s), N(t))$ over $(s,t)$. Let

$$\pi_s^t : N(s) \to N(t)$$

denote parallel translation along $\gamma$.

DEFINITION

A $\gamma$-map $X$ is a smooth map $X:\mathbb{R} \to \mathrm{Hom}\,(N,N)$.

Associated with $X$ are smooth functions $\alpha_X$ and $\omega_X$ such that

$$X(t):N(\alpha(t)) \to N(\omega(t)).$$

If $X$ and $Y$ are $\gamma$-maps with $\alpha_X = \omega_Y$, let $XY$ denote the $\gamma$-map $t \to X(t) \circ Y(t)$.

Suppose $X$ is a $\gamma$-map with $\alpha_X = \alpha$ and $\omega_X = \omega$. Define

$$X(s,t) = \pi_{\omega(s)}^{\omega(t)} \circ X(s) \circ \pi_{\alpha(t)}^{\alpha(s)}.$$

Observe that for any $s$

$$X(s,t) \in \mathrm{Hom}\,(N(\alpha(t)), N(\omega(t))).$$

We see that we can define the *derivative* of $X$ by

$$X'(t) = \frac{d}{ds}\,(X(s,t))\Big|_{s=t}.$$

Given functions $a(t)$ and $b(t)$, we define

$$\int_{a(t)}^{b(t)} X(s)ds = \int_{a(t)}^{b(t)} X(s,t)ds.$$

Note that both of the maps we have just defined are in Hom $(N(\alpha(t)), N(\omega(t)))$. Also define the *adjoint* of $X$

$$X^*(t):N(\omega(t)) \to N(\alpha(t))$$

by

$$\langle X^*(t)v,w\rangle = \langle v,X(t)w\rangle$$

for all $v \in N(\omega(t))$ and $w \in N(\alpha(t))$.

It is easy to show that

$$X'^* = X^{*\prime};$$

$$\int_{a(t)}^{b(t)} X'(s)ds = X(b(t),t) - X(a(t),t);$$

and if $\alpha_X = \omega_Y$

$$(XY)'(t) = \frac{d}{ds}(X(s,t) \circ Y(s,t))\Big|_{s=t}$$

$$= X'Y(t) + XY'(t).$$

We call X *constant* if $X' = 0$ and *symmetric* if $X^* = X$. Note that if X is symmetric, then $\alpha_X = \omega_X$. If X is symmetric, define

$$\lambda^+(X(t)) = \max\{\lambda: \lambda \text{ is an eigenvalue of } X(t)\},$$

$$\lambda^-(X(t)) = \min\{\lambda: \lambda \text{ is an eigenvalue of } X(t)\}.$$

For any $\gamma$-map X define the *upper norm*

$$\|X(t)\| = \sup\{\|X(t)u\| : u \in N(\alpha_X(t)) \text{ and } \|u\| = 1\}$$

and the *lower norm*

$$((X(t))) = \inf\{\|X(t)u\| : u \in N(\alpha_X(t)) \text{ and } \|u\| = 1\}.$$

If $X(t)$ is non-singular,

$$((X(t))) = \|X^{-1}(t)\|^{-1}.$$

If $X(t)$ is symmetric,

$$\|X(t)\| = \sup \{\langle X(t)u,u\rangle : u \in N(\alpha_X(t)) \text{ and } \|u\| = 1\}$$

$$= \max \{|\lambda| : \lambda \text{ is an eigenvalue of } X(t)\},$$

and there is an analogous result for $((X(t)))$.

If $X(t)$ is positive definite, then $\|X(t)\| = \lambda^{+}(X(t))$ and $((X(t))) = \lambda^{-}(X(t))$.

Most of the $\gamma$-maps that we encounter will be of two types:

(i) <u>$\gamma$-maps with $\alpha(t) = t = \omega(t)$</u>.

These are sections of End $(N)$. The derivative defined above is the covariant derivative along $\gamma$.

(ii) <u>$\gamma$-maps with $\alpha(t) = 0$ and $\omega(t) = t$</u>.

*If* $X$ *is such a* $\gamma$*-map and* $v \in N(0)$, *let* $Xv$ *denote the vector field*

$$(Xv)(t) = X(t)(v).$$

Then $X'v$ is the covariant derivative of $Xv$ along $\gamma$.

To help distinguish the two types: script letters will always be of type (i), and $A$, $D$, $J$, $L$ of type (ii).

The curvature tensor defines a $\gamma$-map $\mathcal{K}(t): N(t) \hookleftarrow$ ,

$$R(t)v = R(v,\dot{\gamma}(t))\dot{\gamma}(t).$$

It follows from the symmetrics of the curvature tensor that $R^* = R$.

Remarks

(i) If we choose parallel vector fields that form an orthonormal basis along $\gamma$ (Fermi coordinates), we can represent $\gamma$-maps by $(n-1) \times (n-1)$ matrices. The derivative that we defined above corresponds to differentiating each entry of the matrix. It was in this form that the theory described in this section was first developed by Green [28].

(ii) If we wanted to include tangential Jacobi fields in the discussion, we would replace N by $\gamma^*(TM)$. This is done in matrix form by Pesin in [39, 40, 41].

## B. JACOBI AND LAGRANGE MAPS

Jacobi's equation can be written in terms of $\gamma$-maps.

DEFINITION

A *Jacobi map* along $\gamma$ is a $\gamma$-map $J(t):N(0) \to N(t)$ with the property that $Jv$ is a Jacobi field for every $v \in N(0)$.

The Jacobi maps are the solutions of the *Jacobi map equation*

$$J'' + RJ = 0, \qquad \text{(JM)}$$

where $R$ is the $\gamma$-map defined in §3A.

Clearly a Jacobi map $J$ is uniquely determined by $J(0)$ and $J'(0)$.

3.1 <u>REMARKS</u>

(i) The geodesics $\gamma$ and $\bar{\gamma}$ share the same Jacobi maps (cf. 2.2i).

(ii) We say that a sequence $\{J_n\}$ of Jacobi maps *converges* to the Jacobi map $J$ if there is a sequence $\{s_n\}$ with $s_n \to s$, $J_n(s_n) \to J(s)$ and $J_n'(s_n) \to J'(s)$. Then $J_n(t) \to J(t)$ and $J_n'(t) \to J'(t)$ for any $t$ (uniformly on bounded intervals).

(iii) If $M$ has constant curvature $K$, every Jacobi map has the form

$$J(t) = C_K(t)\, \pi_0^t \circ A + S_K(t) \pi_0^t \circ B$$

where the functions $C_K$ and $S_K$ are defined in Remark 2.2(iii) and $A, B \in \mathrm{End}\,(N(0))$.

(iv) Eschenburg [20, 21] has defined *Jacobi tensors* in almost the same way as we defined Jacobi maps. A Jacobi tensor is a $\gamma$-map $J(t):N(t) \hookleftarrow$ such that $J \circ v$ is a Jacobi field for every parallel section $v$ of $N$. If $J$ is a Jacobi map, $t \to J(t) \circ \pi_t^0$ is a Jacobi tensor. Jacobi tensors have the advantage that all the objects considered in the theory are sections of End $(N)$. On the other hand, they make some results (notably Lemma 5.5) more complicated by introducing extraneous parallel translations which will not appear in our formulation. It should be emphasized that the differences between the two formulations are very minor.

A Jacobi map $J$ is *non-degenerate* if the map $N(0) \to J^{\perp}(\gamma)$, $v \to Jv$, is non-singular. Clearly $J$ is non-degenerate if $J(t)$ is non-singular for some $t$, and degenerate if $\ker J(0) \cap \ker J'(0) \neq \{0\}$.

A Jacobi field $Y$ *belongs* to $J$ if $Y = Jv$ for some $v \in N(0)$. Then $Y'(t) = J'J^{-1}J(t)v = J'J^{-1}(t)Y(t)$ whenever $J(t)$ is non-singular. Conversely, if $J(t_o)$ is non-singular and $Y$ is a Jacobi field with $Y(t_o) = J'J^{-1}(t_o)Y(t_o)$, then $Y$ belongs to $J$. The Jacobi fields belonging to $J$ form a subspace of $J^{\perp}(\gamma)$ which has dimension $n-1$ when $J$ is non-degenerate.

Define the *Wronskian* of two Jacobi maps $J_1$ and $J_2$:

$$W(J_1, J_2) = J_1'{}^*J_2 - J_1^*J_2'.$$

We see that $W(J_1,J_2)(t) \in \mathrm{End}(N(0))$ for all $t$. *The Wronskian is constant.*

$$\begin{aligned} W(J_1,J_2)' &= J_1''{}^*J_2 + J_1'{}^*J_2' - J_1^*{}'J_2' - J_1^*J_2'' \\ &= -(RJ_1)^*J_2 + J_1^* RJ_2 \\ &= -J_1^* R^*J_2 + J_1^* RJ_2 \\ &= 0 \end{aligned}$$

since $R^* = R$. The Wronskian is the analogue for Jacobi maps of the symplectic form $w$ on $J(\gamma)$ defined at the end of §2A. If $v \in N(0)$,

$$\langle W(J_1,J_2)v,v\rangle = w\langle J_1v, J_2v\rangle.$$

We call $J_1$ and $J_2$ *dual* if $W(J_1,J_2) = 0$. A Jacobi map $L$ that is self-dual and non-degenerate is called *Lagrangian* or a *Lagrange tensor.* It can be checked that $L$ is Lagrangian if and only if the Jacobi fields belonging to it form a Lagrangian subspace of $J^{\perp}(\gamma)$. The self-duality of $L$ means that $L^*L'(t):N(0) \hookleftarrow$ is always symmetric and $L'L^{-1}(t):N(t) \hookleftarrow$ is symmetric when $L$ is non-singular. Conversely, if a Jacobi map $L$ has $L(t_o)$ non-singular and $L'L^{-1}(t_o)$ symmetric for some $t_o$, then $L$ is Lagrangian.

A Lagrange map is non-singular except at isolated points. I give Eschenburg's proof.

3.2 <u>PROPOSITION</u> [20, p. 11; 23, Lemma 2]

Suppose $L$ is a Lagrange map with $L(0)$ singular. Then $L(t)$ is non-singular for small nonzero $t$.

<u>Proof</u>

Choose an orthonormal basis $u_1,\ldots,u_{n-1}$ for $N(0)$ so that $u_1,\ldots,u_k$ is a basis for $\ker L(0)$. For $1 < i < n-1$, let $Y_i$ be the Jacobi field $Lu_i$. The result will follow if we show that

$$d(t) = Y_1(t) \wedge \ldots \wedge Y_{n-1}(t) \neq 0$$

for all small enough non-zero $t$. Since $Y_1(0) = \ldots = Y_k(0) = 0$, it is clear that $d(0) = d'(0) = \ldots = d^{(k-1)}(0) = 0$. Thus it will suffice to show that $d^{(k)} \neq 0$.

Now

$$d^{(k)}(0) = Y_1'(0) \wedge \dots \wedge Y_k'(0) \wedge Y_{k+1}(0) \wedge \dots Y_{n-1}(0).$$

The vectors $Y_{k+1}(0),\dots,Y_{n-1}(0)$ are independent, because otherwise some non-zero combination of $u_{k+1},\dots,u_{n-1}$ would be in ker $L(0)$. The vectors $Y_1'(0),\dots,Y_k'(0)$ are also independent because otherwise there would be a non-zero combination $v$ of $u_1,\dots,u_k$ with $L'(0)v = 0$. Since $L(0)v = 0$, we would have $L(t)v \equiv 0$, and $L$ would be degenerate. Finally, since $L$ is self-dual, whenever $1 \leq i \leq k$ and $k + 1 \leq j \leq n-1$, we have

$$\begin{aligned} 0 &= \langle Y_i'(0),\ Y_j(0)\rangle - \langle Y_i(0),\ Y_j'(0)\rangle \\ &= \langle Y_i'(0),\ Y_j(0)\rangle. \end{aligned}$$

It follows that $d^{(k)}(0) \neq 0$. □

Eschenburg [20, §1.4] has given a geometrical characterization of Lagrange maps. I follow the account in [23]. The idea is that a Lagrange map is related to a family of hypersurfaces in a natural way. If $\phi : X \times \mathbb{R} \to M$ we will write

$$\phi(x,t) = \phi_t(x) = \phi^x(t).$$

3.3 <u>DEFINITION</u>

A $C^1$ map $\phi : X \times \mathbb{R} \to M$ is a *normal family of hypersurfaces* along the geodesic $\gamma$ if:

(NF1) $\dim X = n-1$.

(NF2) $\phi^x$ is a geodesic for every $x \in X$, and $\phi^{x_o} = \gamma$ for some $x_o$.

(NF3) For every $(x,t)$,

$$T_x \phi_t : T_x X \to (\dot{\phi}^x)^{\perp}.$$

(NF4) $T_{(x_o,t_o)} \phi$ is non-singular for some $t_o$.

(NF5) $(x,t) \to \dot{\phi}^x(t)$ is $C^1$.

3.4 REMARK

(NF5) actually follows from (NF2) and the $C^1$-ness of $\phi$. The idea is that for small enough $\varepsilon$, $\phi^x$ is the shortest geodesic joining $\phi(x,t)$ and $\phi(x,t+\varepsilon)$. If two points are close enough then the shortest geodesic joining them depends smoothly on them [35, Lemma 10.3]. Since $(x,t) \to \phi(x,t)$ and $(x,t) \to \phi(x,t+\varepsilon)$ are both $C^1$, we see that $(x,t) \to \dot{\phi}^x(t)$ is $C^1$.

Clearly $T_{(x_o,t)}\phi$ is non-singular if and only if $T_{x_o} \phi_t : T_{x_o} X \to N(t)$ is non-singular. For such $t$ there is a neighbourhood $U_t$ of $x_o$ such that $\phi_t(U_t)$ is an embedded hypersurface in $M$. It is $C^2$ since it has a $C^1$ field of unit normals defined by $\phi(x,t) \to \dot{\phi}^x(t)$. Let $II_t$ be the second fundamental tensor of $\phi_t(U_t)$ relative to $\dot{\gamma}(t)$; see §1G for the definition that we use.

3.5 <u>LEMMA</u>

Suppose $w \in T_{x_o}X$ and $Y(t) = T_{x_o}\phi_t(w)$. Then $Y$ is an orthogonal Jacobi field along $\gamma$ and $Y'(t) = II_t(Y(t))$ whenever $T_{(x_o,t)}\phi$ is non-singular.

<u>Proof</u>

Consider a variation of $\gamma$,

$$\alpha(s,t) = \phi(\sigma(s),t),$$

where $\sigma$ is a $C^1$ curve in $X$ with $\sigma(0) = x_o$ and $\dot{\sigma}(0) = w$. Clearly $\alpha$ is $C^1$ and $\frac{\partial\alpha}{\partial s}(0,t) = Y(t)$. We see from (NF3) that $Y(t) \perp \dot{\gamma}(t)$. By (NF5), $\frac{\partial\alpha}{\partial t}$ is $C^1$, and so we can apply (2.5). It follows that $Y(t)$ is a Jacobi field along $\gamma$ and

(0,t). Since $\frac{\partial\alpha}{\partial t}(s,t)$ is always a unit normal to $\phi_t(U_t)$ on the same side as $\dot{\gamma}(t)$, it is clear that $Y'(t) = II_t(Y(t))$ whenever $II_t$ is defined. □

Call a Jacobi field of the above form a $\phi$-*Jacobi field.* A Jacobi map is *related* to $\phi$ if it is non-degenerate and every Jacobi field belonging to it is a $\phi$-Jacobi field.

3.6 <u>LEMMA</u>

Let $J$ be a Jacobi map related to a normal family of hypersurfaces $\phi$. Then $J(t)$ is non-singular if and only if $T_{(x_o,t)}\phi$ is non-singular, and for all such $t$

$$J'J^{-1}(t) = II_t.$$

Proof

We saw above that $T_{(x_o,t)}$ is non-singular if and only if $T_{x_o}\phi_t$ is non-singular. If J is related to $\phi$, $\mathrm{im}(J(t)) \subseteq \mathrm{im}\,(T_{x_o}\phi_t)$. Since both $J(t)$ and $T_{x_o}\phi_t$ are maps between $(n-1)$-dimensional vector spaces, $T_{x_o}\phi_t$ must be non-singular if $J(t)$ is.

Conversely, suppose $T_{x_o}\phi_t$ is non-singular for some t. Then $J'(t) = II_t \circ J(t)$ by (3.5). If $J(t)$ were singular, we would have $\ker J(t) \cap \ker J'(t) = \ker J(t) \neq \{0\}$, and J would be degenerate. Hence $J(t)$ is non-singular and $J'J^{-1}(t) = II_t$. □

Since $II_t$ is always symmetric we see that *any Jacobi map related to* $\phi$ *is Lagrangian*. There is a converse. *Any Lagrange map* L *is related to a normal family of hypersurfaces*. We now construct such a family.

By (3.2) there is $t_o$ with $L(t_o)$ non-singular. Assume for simplicity that $t_o = 0$. Let S be a $C^2$ hypersurface embedded in M, orthogonal to $\gamma$ at $\gamma(0)$. If $p \in S$, let $\hat{p}$ be the unit normal to S on the same side as $\dot{\gamma}(0)$. Let $II(p,\cdot)$ be the second fundamental tensor of S relative to $\hat{p}$. We can and do choose S so that

$$II(\gamma(0),\cdot) = L'L^{-1}(0).$$

Define

$$\phi(p,t) = \gamma_{\hat{p}}(t).$$

Then $\phi$ is a normal family of hypersurfaces. It is clear that $\phi$ satisfies NF1,2,4 and 5. To prove (NF3) it will suffice to show that if $w \in T_pS$, then $Y(t) = T_p\phi_t(w)$ is an orthogonal Jacobi field along $\gamma_{\hat{p}}$. Let $\sigma$ be a $C^1$ curve in S with $\sigma(0) = p$ and $\dot{\sigma}(0) = w$. Consider the $C^1$ variation of $\gamma_{\hat{p}}$ through geodesics

$$\alpha(s,t) = \gamma_{\hat{\sigma}(s)}(t).$$

Clearly $T_p\phi_t(w) = \frac{\partial\alpha}{\partial s}(0,t)$ which is a Jacobi field by (2.5) and is orthogonal by (2.9), since $w \perp \hat{p}$.

Now we show that L is related to $\phi$. In the above we have by (2.5), $Y'(0) = \frac{D}{dt}\frac{\partial\alpha}{\partial s}(0,0) = \frac{D}{ds}\frac{\partial\alpha}{\partial t}(0,0) = II(p,w)$. By taking $p = \gamma(0)$, we see that if Y is a $\phi$-Jacobi field, $Y'(0) = L'L^{-1}(0)(Y(0))$, and so Y belongs to L.

Many of the later geometrical results will be based on the study of three Lagrange maps and their related hypersurfaces. The first of these is introduced in the next paragraph, and the others in §5.

## C. THE LAGRANGE MAP A

Let $A_a$ be the Jacobi map along $\gamma$ defined by $A_a(a) = 0$ and $A_a'(a) = I$. Here, and elsewhere, $I:N(0) \to N(0)$ is the identity map. Clearly $W(A_a,A_a)(a) = 0$, so $A_a$ is a Lagrange map. Write $A = A_0$. The Jacobi fields belonging to A are the initially vanishing orthogonal Jacobi fields. If Y is such a Jacobi field,

$$Y(t) = A(t)Y'(0).$$

A is important because it is related to the geodesic spheres $S(\gamma(0),t)$ with centre $\gamma(0)$. Define $e:S_{\gamma(0)}M \times \mathbb{R} \to M$ by $e(u,t) = \exp_{\gamma(0)} tu$. In the notation of §3B, $S(\gamma(0),t)$ is the image under $e_t$ of $S_{\gamma(0)}M$, and the curves $e^u$ are the geodesics through $\gamma(0)$. It is clear from Gauss' Lemma (1.2) that e is a normal family of hypersurfaces along $\gamma$, and the e-Jacobi fields are the initially vanishing orthogonal Jacobi fields. Thus A is related to e.

From this and (3.6) we obtain

3.7 PROPOSITION

$A(t)$ is non-singular if and only if $T_{t\dot\gamma(0)}\exp_{\gamma(0)}$ is non-singular, and for all such t the second fundamental tensor of $S(\gamma(0),t)$ relative to $\dot\gamma(t)$ is $A'A^{-1}(t)$.

For small t, $A(t)$ and $A'A^{-1}(t)$ behave as they do in Euclidean space.

3.8 LEMMA

For any geodesic, $\lim_{t\to 0} t^{-1} A(t) = I = \lim_{t\to 0} tA'A(t)$.

Proof

$\lim_{t\to 0} t^{-1}A(t) = I$ follows from $A(0) = 0$ and $A'(0) = I$.
Using this,

$$\lim_{t\to 0} t\, A'A^{-1}(0) = \lim_{t\to 0} A'(0) \lim_{t\to 0} [t^{-1}A(t)]^{-1} = I. \qquad \square$$

This leads to a test for detecting singularities of Lagrange maps.

3.9 <u>COROLLARY</u>

Suppose L is a Lagrange map with L(0) singular. Then

$$\lim_{t\to 0^+} \lambda^+(L(t)) = \infty$$

and

$$\lim_{t\to 0^-} \lambda^-(L(t)) = -\infty.$$

For the definition of $\lambda^+$ and $\lambda^-$ see §3A.

<u>Proof</u>

Let Y be a non-zero Jacobi field belonging to L which has Y(0) = 0. Clearly Y(t) ≠ 0 for small non-zero t. Since Y(0) = 0, Y belongs to A as well as L. By (3.2), both A(t) and L(t) are non-singular for small t ≠ 0, and so for such t, $L'L^{-1}(t)Y(t) = Y'(t) = A'A^{-1}(t)Y(t)$. Since $\lim_{t\to 0} t\, A'A^{-1}(t) = I$, it follows that

$$\lim_{t\to 0} t\left\langle L'L^{-1}(t) \frac{Y(t)}{\|Y(t)\|}, \frac{Y(t)}{\|Y(t)\|}\right\rangle = 1,$$

from which the result is obvious. $\square$

## D. A RICCATI EQUATION

Suppose $J$ is a Jacobi map along $\gamma$. When $J(t)$ is non-singular, let $U(t) = J'J^{-1}(t)$. Then $U(t) \in \mathrm{End}(N(t))$, and if $Y$ is a Jacobi field belonging to $J$,

$$Y'(t) = U(t)Y(t).$$

We have

$$\begin{aligned} U'(t) &= J''J^{-1}(t) + J'(J^{-1})'(t) \\ &= -RJJ^{-1}(t) - J'J^{-1}J'J^{-1}(t) \\ &= -R - U^2(t). \end{aligned}$$

Thus $U$ satisfies the *Riccati equation*

$$U' + U^2 + R = 0. \qquad \text{(R)}$$

(Note that all the terms appearing here are sections of $\mathrm{End}(N(t))$.)

Conversely, if $U$ is a solution of (R) defined on the interval $(\alpha,\beta)$ then $U = J'J^{-1}$ for some Jacobi map $J$ that is non-singular on $(\alpha,\beta)$: solving the equation $J' = UJ$ gives us a family of such Jacobi maps which differ only by a constant factor. We saw in §3B that $U$ is symmetric if and only if $J$ is a Lagrange map, and then $U$ gives the second fundamental forms of the hypersurfaces related to $J$.

We will prove some comparison results for symmetric solutions of (R). They will be used later when we study

Jacobi fields in manifolds with no conjugate and no focal points.

If $U_1$ and $U_2$ are symmetric endomorphisms of an inner product space, write $U_1 > U_2$ (resp. $\geq$, $<$, $\leq$) if $U_1 - U_2$ is positive definite (resp. positive semi-definite, negative definite, negative semi-definite).

3.10 <u>LEMMA</u> [12, p.50]

Let $U_1$ and $U_2$ be symmetric solutions of the Riccati equation (R) on the (possibly unbounded) interval $(\alpha,\beta)$. If $U_1(a) \geq U_2(a)$ for some $a \in (\alpha,\beta)$, then $U_1(t) \geq U_2(t)$ for all $t \in (\alpha,\beta)$. If $U_1(a) > U_2(a)$ then $U_1(t) > U_2(t)$ for all $t \in (\alpha,\beta)$.

<u>Proof</u>

Write $D(t) = U_1(t) - U_2(t)$ and $M(t) = \frac{1}{2}\{U_1(t) + U_2(t)\}$. Then $D(t)$ is symmetric and

$$D' + DM + MD = 0.$$

Let $X(t): N(t) \to N(t)$ be a solution of the linear equation

$$X' = MX$$

with $X(a)$ non-singular. Then $X(t)$ is non-singular for all $t \in (\alpha,\beta)$ and, since $M$ is symmetric

$$(X^*DX)' = X^*(D' + DM + MD)X = 0.$$

Thus $X^*DX(t)$ is constant. It follows that the signature of $D(t)$ is constant. □

3.11 LEMMA

Suppose $\mathcal{U}$ is a symmetric solution of the Riccati equation (R) on the interval $(\alpha,\beta)$, and B is a Lagrange map with B(b) non-singular for some $b \in (\alpha,\beta)$. If $B'B^{-1}(b) \prec \mathcal{U}(b)$, then B(t) is non-singular for $\alpha < t \leq b$. If $B'B^{-1}(b) \succ \mathcal{U}(b)$, then B(t) is non-singular for $b \leq t < \beta$.

Proof

We prove the first claim. Let $a = \sup\{t: \alpha < t < b$ and B(t) is singular$\}$. By (3.10), $B'B^{-1}(t) \prec \mathcal{U}(t)$ for $a < t < b$.

Suppose $a > \alpha$. Then B(a) is singular. By (3.9), $\lim_{t \to a^+} \lambda^+(B(t)) = \infty$, which contradicts the above inequality. □

The next two propositions involve a Lagrange map L that is non-singular on an interval. The first compares L with the Lagrange maps that vanish at the endpoints.

3.12 PROPOSITION

Suppose the Lagrange map L(t) is non-singular for $a < t < b$. Then $A_a(t)$ and $A_b(t)$ are both non-singular for $a < t < b$, and for such t

$$A_a'A_a^{-1}(t) \succ L'L^{-1}(t) \succ A_b'A_b^{-1}(t).$$

These inequalities are both strict if L(t) is also non-singular at t = a and t = b.

Proof

We prove the inequality for $A_a$. The inequality for $A_b$ then follows by thinking of L as a Lagrange map along $\bar{\gamma}$.

Assume for simplicity that $a = 0$.

First we show that if $0 < s < b$, then $A_s(t)$ is non-singular and $A_s'A_s^{-1}(t) > L'L^{-1}(t)$ for $s < t < b$. Since L is non-singular on $(0,b)$, $L'L^{-1}$ is continuous at s. We see from (3.8) that for small enough $\varepsilon > 0$, $A_s'A_s^{-1}(s+\varepsilon) > L'L^{-1}(s+\varepsilon)$. It follows from (3.11) (with $U = L'L^{-1}$) and (3.10) that $A_s(t)$ is non-singular and $A_s'A_s^{-1}(t) > L'L^{-1}(t)$ for $s + \varepsilon < t < b$. Letting $\varepsilon \to 0$ proves the above claim about $A_s$.

Clearly $A_s \to A$ as $s \to 0$. We see from (3.1ii) that $A_s'A_s^{-1}(t) \to A'A^{-1}(t)$ whenever $A(t)$ is non-singular. It follows that $A'A^{-1}(t) \geq L'L^{-1}(t)$ whenever $0 < t < b$ and $A(t)$ is non-singular. Now we show that $A(t)$ is non-singular for $0 < t < b$. By (3.2), $A(t)$ is non-singular for any small enough $\varepsilon > 0$. It follows from the above inequality and (3.11) (with $U = L'L^{-1}$) that $A(t)$ is non-singular for $\varepsilon < t < b$. Now let $\varepsilon \to 0$.

Finally, if $L(t)$ is non-singular for $0 < t < b$, there are $a' < 0$ and $b' > b$ such that $L(t)$ is non-singular for $a' < t < b'$. The argument of the third paragraph shows that $A(t)$ is non-singular and $A'A^{-1}(t) > L'L^{-1}(t)$ for $0 < t < b'$. □

REMARK

It is essential in the above that L be a Lagrange map. For example, let $\gamma$ be a great circle in $S^3$ with the metric of curvature 1. Choose u and v so that $\{\dot{\gamma}(0),u,v\}$ is an orthonormal basis of $T_{\gamma(0)}S^3$. The Jacobi map J defined by

$$J(t)u = \cos t u + \sin t v,$$

$$J(t)v = -\sin t u + \cos t v$$

is never singular, even though $A(n\pi)$ is singular for every integer n. To visualize J, think of $\gamma$ as a fibre of the Hopf map $S^3 \to S^2$. The Jacobi fields obtained by varying $\gamma$ through fibres will belong to J.

The second proposition considers the effect on L of a curvature bound on M.

3.13 <u>PROPOSITION</u> [17, Proposition 2.7]

Suppose that the Lagrange map L(t) is non-singular for $a < t < b$. Suppose $-k^2$ is a lower bound for the sectional curvature of any plane containing $\dot{\gamma}(t)$ for $a < t < b$. Then for $a < t < b$,

$$k \coth k(t-b)\, I(t) \leq L'L^{-1}(t) \leq k \coth k(t-a)\, I(t),$$

where $I(t):N(t) \to N(t)$ is the identity map.

The left and right hand terms are the solutions of $U' + U^2 - k^2 I = 0$ corresponding to the Lagrange maps $A_a$ and $A_b$ along a geodesic in a manifold with constant curvature $-k^2$.

<u>Proof</u>

We prove the right hand inequality. The left hand one then follows by thinking of L as a Lagrange map along $\bar{\gamma}$.

Write $\mathcal{U} = L'L^{-1}$. It will be enough to show that for any parallel, orthogonal unit vector field u along $\gamma$,

$$f(t) = \langle \mathcal{U}(t)u(t), u(t)\rangle \leq k \coth k(t-a)$$

for $a < t < b$. It follows from the Riccati equation (R) that

$$\langle \mathcal{U}'(t)u(t), u(t)\rangle + \langle \mathcal{U}(t)^2 u(t), u(t)\rangle + \langle \mathcal{R}(t)u(t), u(t)\rangle = 0$$

for $a < t < b$. Since u is parallel, $\langle \mathcal{U}'(t)u(t), u(t)\rangle = f'(t)$. Since $\mathcal{U}(t)$ is symmetric, $\langle \mathcal{U}(t)^2 u(t), u(t)\rangle = \|\mathcal{U}(t)u(t)\|^2 \geq f(t)^2$. Since it is the sectional curvature of the plane spanned by $\dot{\gamma}(t)$ and $u(t)$, $\langle \mathcal{R}(t)u(t), u(t)\rangle \geq -k^2$. Thus f(t) satisfies

$$f'(t) + f(t)^2 - k^2 \leq 0$$

for $a < t < b$. For each $s \in (a,b)$, $g_s(t) = k \coth k(t-s)$ is a solution of

$$g'(t) + g(t)^2 - k^2 = 0$$

and $g_s(t) \to \infty$ as $t \to s^+$. Since f is continuous on $(a,b)$, we have $g_s(s+\varepsilon) > f(s+\varepsilon)$ for any $s \in (a,b)$ and for all small enough $\varepsilon > 0$. It follows, using (3.14) below that $f(t) \leq g_s(t)$ for $s < t < b$. Since $g_s(t) \to k \coth k(t-a)$ as $s \to a$ we are done. □

3.14 LEMMA

Suppose $k(t)$ is a continuous function and f(t) and g(t) satisfy

$$f'(t) + f(t)^2 + k(t) < 0,$$

$$g'(t) + g(t)^2 + k(t) = 0,$$

$$g(t) > 0$$

for $c < t < d$. If $f(c) < g(c)$, then $f(t) < g(t)$ for $c < t < d$.

<u>Proof</u>

Suppose not. Then there is $s \in (c,d)$ with $f(s) - g(s) > 0$. Let $r$ be the largest number less than $s$ with $f(r) - g(r) < 0$. Then $c < r < s$. For $r < t < s$, we have $f(t) > g(t) > 0$, and so for such $t$

$$\begin{aligned} f'(t) - g'(t) &= f'(t) + g(t)^2 + k(t) \\ &< g(t)^2 - f(t)^2 < 0. \end{aligned}$$

Hence $f(s) - g(s) < f(r) - g(r) < 0$, contrary to the choice of $s$. □

## E. <u>REDUCTION OF ORDER</u>

As well as using the comparison theorems of §3D we will need to be able to express one Lagrange map in terms of another. Suppose $L$ is a Lagrange map such that $L(t)$ is non-singular for all $t$ in the (possibly unbounded) interval $(\alpha,\beta)$. The method of reduction of order enables us to express any Jacobi map in terms of $L$ along $(\alpha,\beta)$.

3.15 <u>LEMMA</u>

A $\gamma$-map $J$ satisfies the Jacobi map equation (JM) for $t \in (\alpha,\beta)$ if and only if $W(J,L)(t)$ is constant for $t \in (\alpha,\beta)$.

<u>Proof</u>

We already know the only if result from §3B. Conversely, if $W(J,L)$ is constant,

$$\begin{aligned} 0 &= (J'^*L - J^*L')' \\ &= J''^*L - J^*L'' \\ &= J''^*L + J^*RL \\ &= (J'' + RJ)^*L, \text{ since } R^* = R. \end{aligned}$$

Since $L(t)$ is non-singular for $t \in (\alpha,\beta)$ we are done. □

Now we solve the equation $W(J,L)$ = constant. Suppose

$$W(J,L)(t) \equiv B \in \mathrm{End}(N(0))$$

for $t \in (\alpha,\beta)$. Put $J = LX$, where $X(t):N(0) \to N(0)$. Then

$$\begin{aligned} -B &= W(L,J) \\ &= L'^*LX - L^*(LX)' \\ &= (L'^*L - L^*L')X - L^*LX' \\ &= -L^*LX', \end{aligned}$$

since $L$ is Lagrangian. Thus $X' = (L^*L)^{-1}B$. From this and the above lemma, we easily obtain

3.16 <u>PROPOSITION</u> [22, Proposition 2]

Suppose $J$ is a Jacobi map. Given $s \in (\alpha,\beta)$ there are $B$, $C \in \mathrm{End}\,(N(0))$ such that

$$J(t) = L(t)\ \{\int_s^t (L^*L)^{-1}(x)\,dx\ B + C\},$$

for any $t \in (\alpha,\beta)$. In fact

$$B = W(J,L) \text{ and } C = L^{-1}J(s).$$

Conversely, any $\gamma$-map $J$ of the above form is a Jacobi map with $W(J,L) = B$.

Note that the integrand $(L^*L)^{-1}$ is symmetric and positive definite. An analogous formula can be derived even when $L$ is not a Lagrange map by writing $J = KX$, where $K$ is a Jacobi map dual to $L$. However, the integrand corresponding to $(L^*L)^{-1}$ is not symmetric.

## F. <u>FIELDS OF JACOBI MAPS</u>

So far we have considered Jacobi maps along a single geodesic $\gamma$. Later we will also encounter *fields of Jacobi maps* which assign to each $v \in SM$ a Jacobi map along $\gamma_v$. To illustrate the notation we will use: $A(v,t)$ is a field of Jacobi maps, with $A(\dot{\gamma}(0),t)$ the Jacobi map $A$ along $\gamma$ defined in §3C. The dependence on $v$ will be suppressed whenever possible.

A field $J$ of Jacobi maps is *continuous* if $(v,t) \to J(v,t)$ is continuous. We see that $J$ is continuous if and only if both $v \to J(v,0)$ and $v \to J'(v,0)$ are continuous. If $J$ is continuous so is $(v,t) \to J'(v,t)$. These properties follow because Jacobi maps satisfy the differential equation (JM).

Example: $A(v,t)$ is continuous.

## §4. METRICS WITH NO CONJUGATE AND NO FOCAL POINTS

This section introduces the class of Riemannian metrics - those with no focal points - that will be studied in the remainder of this thesis. The theory described there was originally developed for manifolds with non-positive curvature [19, 4, 5]. Later work [20, 21, 22, 25, 26, 39] has made it possible to extend the results to a wider class of metrics.

The theory requires the study of geodesics and spheres of large radius in the universal cover. For the constructions involved one needs the property that any two points are joined by a unique geodesic. We will see (4.8) that this is equivalent to a condition on Jacobi fields: namely that there be no conjugate points.

It has not, however, been possible to construct a satisfactory theory without assuming the stronger condition that the metric have no focal points. Geometrically, this means assuming that geodesic balls in the universal cover are convex (4.14).

### A. CONJUGATE POINTS

The points $\gamma(a)$ and $\gamma(b)$ are *conjugate* along the geodesic $\gamma$ if $a \neq b$ and there is a non-trivial Jacobi field $Y$ along $\gamma$ such that $Y(a) = 0$ and $Y(b) = 0$. The *multiplicity* with which they are conjugate is the number of independent Jacobi fields with these properties.

REMARKS

(i) This is ambiguous if $\gamma$ intersects itself. Strictly, it should be a and b that are conjugate.

(ii) It is clear from (2.7) that the Jacobi field Y above must be orthogonal.

Example

If $\gamma$ is a great circle on the unit sphere $S^n$ $(n > 2)$, $\gamma(0)$ and $\gamma(k\pi)$ are conjugate with multiplicity n-1 for any $k \neq 0$.

We will use Lagrange maps to study conjugacy.

4.1 PROPOSITION

If $a \neq b$ the following are equivalent.

(i) $\gamma(a)$ and $\gamma(b)$ are conjugate.

(ii) $A_a(b)$ is singular.

(iii) $\exp_{\gamma(a)}$ is singular at $(b-a)\dot{\gamma}(a)$.

Proof

(i) $\Leftrightarrow$ (ii). Obvious. (ii) $\Leftrightarrow$ (iii). Use (3.7). □

It follows from (ii) and (3.2) that the points along $\gamma$ conjugate to $\gamma(a)$ are isolated.

We now present two lemmas to detect the presence and absence of conjugate points.

4.2 LEMMA

Suppose that $a < 0 < b$ and $\gamma(0)$ is not conjugate to $\gamma(a)$ or $\gamma(b)$. Then $\gamma(a)$ and $\gamma(b)$ are conjugate if and only if $D = A_b'A_b^{-1}(0) - A_a'A_a^{-1}(0)$ is singular.

This means that $\gamma(a)$ and $\gamma(b)$ are conjugate when the spheres $S(\gamma(a), -a)$ and $S(\gamma(b), b)$ make second order contact at $\gamma(0)$.

Proof

If $D$ is singular, there is a non-zero $v \in \ker(D)$. Since $\gamma(a)$ is not conjugate to $\gamma(0)$, $A_a(0)$ is non-singular by (4.1), and so there is a Jacobi field $Y$ belonging to $A_a$ with $Y(0) = v$. Then $Y'(0) = A_a'A_a^{-1}(0)(v) = A_b'A_b^{-1}(0)(v)$ since $v \in \ker(D)$. Thus $Y$ also belongs to $A_b$. It follows that $Y(a) = 0$ and $Y(b) = 0$, and so $Y(a)$ and $Y(b)$ are conjugate.

Conversely, if $\gamma(a)$ and $\gamma(b)$ are conjugate, there is an orthogonal Jacobi field $Y$ with $Y(a) = 0$, $Y(b) = 0$ and $Y(0) \neq 0$. Since $Y$ belongs to both $A_a$ and $A_b$, $A_a'A_a(0)(Y(0)) = Y'(0) = A_b'A_b^{-1}(0)(Y(0))$, and so $D(Y(0)) = 0$. $\square$

4.3 LEMMA

Suppose the Lagrange tensor $L(t)$ is non-singular for $a < t < b$. Then the only points of $\gamma|[a,b]$ which can be conjugate are $\gamma(a)$ and $\gamma(b)$.

Proof

We use (4.1). By (3.12), $A_a(t)$ and $A_b(t)$ are non-singular for $a < t < b$, and so, for such $t$, $\gamma(t)$ is not conjugate to $\gamma(a)$ or $\gamma(b)$. Similarly, if $a < s < t < b$, $A_s(t)$ is non-singular and so $\gamma(s)$ is not conjugate to $\gamma(b)$. □

If $I \subseteq \mathbb{R}$ is an interval, the geodesic segment $\gamma|I$ *has no conjugate points* if $\gamma(s)$ and $\gamma(t)$ are not conjugate for any $s,t \in I$.

4.4 COROLLARY

Suppose $a < b$. The following are equivalent.

(i) $\gamma|(a,b)$ has no conjugate points.

(ii) $\gamma(a)$ is not conjugate to $\gamma(t)$ for $a < t < b$.

(iii) $\gamma(b)$ is not conjugate to $\gamma(t)$ for $a < t < b$.

(iv) $\gamma(r)$ is not conjugate to $\gamma(s)$ if $a < r < b$ and $a < s < b$.

Proof

(i) $\Rightarrow$ (iii). Choose $r$ so that $a < r < t$. Since $\gamma(r)$ is not conjugate to $\gamma(s)$ for $r < s < b$, $A_r(s)$ is non-singular for $r < s < b$. By (4.3), $\gamma(b)$ is not conjugate to any $\gamma(s)$ with $r < s < b$, in particular not to $\gamma(t)$.

(iii) $\Rightarrow$ (ii). $A_b(t)$ is non-singular for $a < t < b$. Apply (4.3) with $L = A_b$.

(ii) $\Rightarrow$ (iv). $A_a(t)$ is non-singular for $a < t < b$. Apply (4.3) with $L = A_a$.

(iv) $\Rightarrow$ (i). Trivial. □

Conjugate points are related to the index of a geodesic segment when it is considered as a critical point of the energy function on curves joining its endpoints. The Morse Index Theorem [35, Theorem 15.1] states that the index of $\gamma|[a,b]$ is the number of points of $\gamma|[a,b]$ conjugate to $\gamma(a)$ counted according to multiplicity. We prove a simpler result.

4.5 <u>PROPOSITION</u>

Suppose $a < b$ and $\gamma(a)$ is conjugate to $\gamma(b)$. Then $\gamma|[a,s]$ is not a minimal geodesic for any $s > b$.

<u>Proof</u>

If $\gamma|[a,s]$ is not minimal, neither is $\gamma|[a,s']$ for any $s' > s$. Thus we can assume that $\gamma(a)$ is not conjugate to $\gamma(t)$ for $a < t < b$, and need only consider values of $s$ slightly bigger than $b$. By reparametrizing if necessary, we can assume that $a < 0 < b$. By (4.4iv), $\gamma(0)$ is not conjugate to $\gamma(t)$ for $a < t < b$. Clearly there is $c > b$ such that $\gamma(0)$ is not conjugate to $\gamma(t)$ for $a < t < c$. We will show that if $b < s < c$ the open geodesic balls $B(\gamma(a), -a)$ and $B(\gamma(s),s)$ intersect, which proves that $\gamma|[a,s]$ is not minimal. To do this, we compare the second fundamental tensors $II^a$ and $II^s$ of $S(\gamma(a), -a)$ and $S(\gamma(s),s)$ respectively relative to their common normal $\dot\gamma(0)$. Since $\dot\gamma(0)$ points into $B(\gamma(s),s)$, $B(\gamma(a),-a)$ and $B(\gamma(s),s)$ will intersect if $II^s - II^a$ has a positive eigenvalue (see §1G).

Clearly $A_a(0)$ and $A_s(0)$ (for $0 < s < c$) are non-singular. By (3.7) we have

$$II^s - II^a = A_s'A_s^{-1}(0) - A_a'A_a^{-1}(0)$$

for $0 < s < c$. By (4.2), this has 0 as an eigenvalue when $s = b$. It follows immediately from (4.6) below that $II^s - II^a$ has a positive eigenvalue for $b < s < c$. □

4.6 LEMMA

Suppose $\gamma(0)$ is not conjugate to $\gamma(t)$ for $0 < t < c$. If $0 < r < s < c$,

$$A_r'A_r^{-1}(0) < A_s'A_s^{-1}(0).$$

Proof

By (4.4), $\gamma(s)$ is not conjugate to $\gamma(t)$ for $0 \leq t < c$. Hence $A_s(t)$ is non-singular for $0 \leq t < s$. It follows that there is $a < 0$ such that $A_s(t)$ is non-singular for $a \leq t \leq r$. Proposition (3.12) (with $L = A_s$) tells us that $A_r(t)$ is non-singular and $A_s'A_s^{-1}(t) > A_r'A_r^{-1}(t)$ for $a < t < r$; in particular for $t = 0$. □

## B. METRICS WITH NO CONJUGATE POINTS

A Riemannian manifold has *no conjugate points* if no two points are conjugate along any geodesic. It is clear from (3.1iii) that manifolds with constant curvature $\leq 0$ have no conjugate points. We will see in §4C that any manifold with curvature $\leq 0$ has no conjugate points. We obtain the following characterization of manifolds with no conjugate points from (4.1).

4.7 PROPOSITION

A Riemannian manifold M has no conjugate points if and only if the exponential map $\exp_p$ is a local diffeomorphism for every $p \in M$.

4.8 PROPOSITION

Let H be a simply connected Riemannian manifold. Then the following are equivalent.

(i) H has no conjugate points.

(ii) $\exp_p : T_pH \to H$ is a diffeomorphism for every $p \in H$.

(iii) There is a unique (up to reparametrization) geodesic passing through any two distinct points of H.

(iv) Every geodesic segment in H is minimal: if $\gamma$ is a geodesic, $d(\gamma(s), \gamma(t)) = |t-s|$.

Proof

(i) $\Rightarrow$ (ii). Use (4.7). (ii) $\Rightarrow$ (iii). Obvious. (iii) $\Rightarrow$ (iv). By the Hopf-Rinow theorem, any two points are joined by a minimal geodesic. Thus, if there is only one geodesic joining two points, it is minimal. (iv) $\Rightarrow$ (i). If H has conjugate points, there will be non-minimal geodesic segments by (4.5). □

Part (iii) of the above will be used extensively. It allows the following definitions.

4.9 DEFINITION

Let H be simply connected with no conjugate points. If p and q are distinct points in H, let $\gamma_{pq}$ be the geodesic with

$$\gamma_{pq}(0) = p \text{ and } \gamma_{pq}(d(p,q)) = q.$$

Let $V(p,q) = \dot{\gamma}_{pq}(0)$. If $v \in T_pM \setminus \{0\}$, $\measuredangle_p(v,q) = \measuredangle_p(v,V(p,q))$. If $q_1 \neq p \neq q_2$, $\measuredangle_p(q_1,q_2) = \measuredangle_p(V(p,q_1),V(p,q_2))$. If $A \subseteq H$, $\measuredangle_p(A) = \sup\{\measuredangle_p(q_1,q_2) : q_1,q_2 \in A \setminus \{p\}\}$.

We will often study the function which gives distance from a fixed point $p_o$.

4.10 LEMMA

Let H be as in (4.9). Let $p_o \in H$ and write $f = d(p_o,\cdot)$. Define the vector field on $H \setminus \{p_o\}$, $W = -V(\cdot,p_o)$. Suppose $p \in H \setminus \{p_o\}$.

(i) f is smooth at p and $\operatorname{grad}_p f = W(p)$.

(ii) $\nabla_{W(p)} W = 0$, and for $v \in W(p)^{\perp}$

the map $v \to \nabla_v W$ is the second fundamental tensor of the sphere $S(p_o, f(p))$ relative to the outward unit normal $W(p)$.

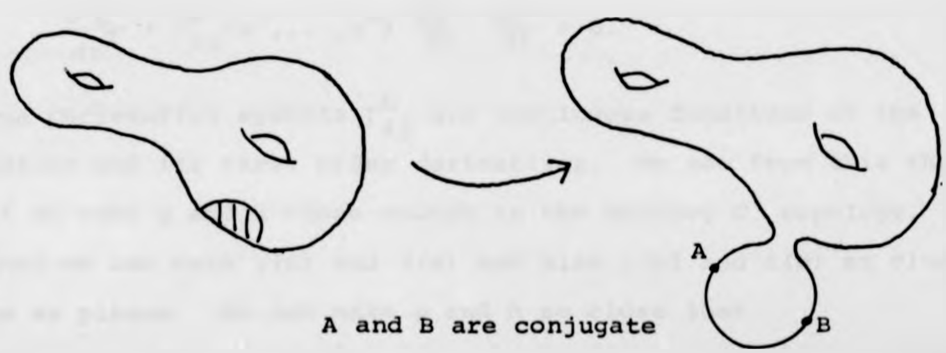
Proof

Write $\gamma = \gamma_{p_o p}$.

(i) Since $\exp_{p_o}$ is a diffeomorphism by (4.8ii), it is clear from (4.8iv) that $f = \|\exp_{p_o}^{-1}(\cdot)\|$, which is smooth except at $p_o$. It is clear from the triangle inequality that $\|\text{grad } f\| \leq 1$. Since $f(\gamma(t)) = |t|$ and $p = \gamma(f(p))$, we see that $\text{grad}_p f = \dot{\gamma}(f(p)) = W(p)$.

(ii) $\nabla_{W(p)} W = 0$ is clear, since $W(\gamma(t)) = \dot{\gamma}(t)$ for $t > 0$. The other claim follows because, on $S(p_o, f(p))$, $W$ is the field of outward unit normals. $\square$

We see from (4.8ii) that the universal cover of a manifold M which admits a metric with no conjugate points is diffeomorphic to $\mathbb{R}^n$. In particular, $\pi_k(M) = 0$ for $k \geq 2$. This means that on many manifolds - spheres for example - there is no metric without conjugate points. On the other hand every manifold has metrics with conjugate points. If we deform a small disc until it is almost a sphere, antipodal points on this sphere will be conjugate.




A and B are conjugate

We can show that having no conjugate points is a closed condition on Riemannian metrics in the following sense.

4.11 PROPOSITION

Let M be a manifold. The set of Riemannian metrics on M with conjugate points is open in the Whitney $C^1$ topology on the space $\mathcal{R}(M)$ of smooth, symmetric, positive definite sections of $T^*M \otimes T^*M$. (The definition of this topology is clear from pp. 531-532 of [1]).

Proof

Let H be the universal cover of M. Suppose g is a metric on M with conjugate points. Then its lift to H (again denoted by g) has conjugate points, and so by (4.8) there is a g-geodesic $\gamma$ in H which is not minimal. Thus there are a and b with $d_g(\gamma(a), \gamma(b)) < |b-a|$.

Now suppose that h is another metric on M. Let $\delta$ be the h-geodesic with $\dot{\delta}(0) = \|\dot{\gamma}(0)\|_h^{-1} \dot{\gamma}(0)$. In local coordinates the geodesic equation is

$$\frac{d^2u^k}{dt^2} + \Gamma^k_{ij}(u^1,\ldots,u^n) \frac{du^i}{dt} \frac{du^j}{dt} = 0.$$

The Christoffel symbols $\Gamma^k_{ij}$ are continuous functions of the metric and its first order derivatives. We see from this that if we make g and h close enough in the Whitney $C^1$ topology, then we can make $\gamma(a)$ and $\delta(a)$ and also $\gamma(b)$ and $\delta(b)$ as close as we please. We can make g and h so close that

$d_h(\delta(a),\ \delta(b)) < |b-a|$. It follows from (4.8) that h has conjugate points on H and hence also on M. □

A similar argument shows that if we fix a metric on M, then $\{v \in SM: \gamma_v(a)$ and $\gamma_v(b)$ are conjugate along $\gamma_v$ for some a and b$\}$ is open; this fact is stated on p.7 of [33].

There are two main classes of manifold that are known to have no conjugate points. The first is compact manifolds whose geodesic flow is Anosov; see [33]. The second is manifolds with no focal points, which we consider next. See especially (4.12) and the subsequent discussion.

## C. METRICS WITH NO FOCAL POINTS

We say that $\gamma(a)$ is a *focal point* of $\gamma(b)$ along the geodesic $\gamma$ if there is a Jacobi field Y such that $Y(a) = 0$, $Y(b) \neq 0$ and $\langle Y,Y\rangle'(b) = 0$. Note that this relation, unlike conjugacy, is not symmetric. It is clear from (2.7) that Y must be orthogonal. If $\gamma$ has self intersections there is the same ambiguity as for conjugacy.

The usual definition of a focal point is that a sub-manifold S which is orthogonal to $\dot{\gamma}(b)$ has a focal point at $\gamma(a)$ if there is a Jacobi field, obtained by varying Y through geodesics orthogonal to S, which vanishes at a but not at b. This is related to our definition as follows. Let $\delta$ be the (non unit speed) geodesic with $\dot{\delta}(0) = Y(b)$. Then, according to the usual definition, $\delta$ has a focal point at $\gamma(a)$. To prove this, we show that Y can be obtained by varying $\gamma$ in the above manner. Since $\dot{\gamma}(b)$, $Y'(b)$ and $\dot{\delta}(0) = Y(b)$ are all orthogonal to one another, there is a $C^\infty$ unit vector field $u$ along $\delta$ with $u(0) = \dot{\gamma}(b)$, $D_\delta u(0) = Y'(b)$ and $u(s) \perp \dot{\delta}(s)$ for all $s$. Then $\alpha(s,t) = \gamma_{u(s)}(t)$ is a $C^\infty$ variation of $\gamma$ through geodesics orthogonal to $\delta$ and $Y = \frac{\partial\alpha}{\partial s}(0,\cdot)$, since $Y(b) = \frac{\partial\alpha}{\partial s}(0,b)$ and $Y'(b) = \frac{D}{dt}\frac{\partial\alpha}{\partial s}(0,b)$.

A Riemannian manifold has *no focal points* if no point is a focal point of another along any geodesic.

4.12 PROPOSITION

Each of the following properties of a Riemannian manifold $M$ implies the following ones.

(i) $M$ has sectional curvature $\leq 0$.

(ii) $M$ has no focal points.

(iii) $M$ has no conjugate points.

Proof

(i) $\Rightarrow$ (ii). If $Y$ is any Jacobi field along a geodesic $\gamma$ in a manifold of non-positive curvature, $\langle Y,Y\rangle(t)$ is a convex function of $t$. For

$$
\begin{aligned}
\tfrac{1}{2}\langle Y,Y\rangle'' &= \langle Y',Y\rangle' \\
&= \langle Y'',Y\rangle + \langle Y',Y'\rangle \\
&= -\langle R(Y,\dot{\gamma})\dot{\gamma},Y\rangle + \langle Y',Y'\rangle \\
&\geq 0.
\end{aligned}
$$

Now suppose that $Y$ is a non-trivial Jacobi field with $Y(a) = 0$. Then $\langle Y,Y\rangle'(a) = 0$ and $\langle Y,Y\rangle''(a) = 2\langle Y',Y'\rangle(a) \neq 0$. It follows from the convexity property that if $b \neq a$, then $\langle Y,Y\rangle'(b) \neq 0$ and so $\gamma(a)$ is not a focal point of $\gamma(b)$.

(ii) $\Rightarrow$ (iii). Suppose $M$ has conjugate points. Then there are $a < b$ and a non-trivial Jacobi field $Y$ along a geodesic $\gamma$ with $Y(a) = 0$ and $Y(b) = 0$. Choose $c \in (a,b)$ with $\|Y(c)\|$ as large as possible. Then $\gamma(a)$ and $\gamma(b)$ are both focal points of $\gamma(c)$. □

Gulliver [30] has constructed examples to show that the converses of both the above implications are false. His simplest examples are 'formed by "raising a blister" on compact manifolds of constant negative curvature'. He also has an example of a manifold with focal points and Anosov geodesic flow. Eberlein [17, 18] and Bolton [9] have given conditions for a manifold with no conjugate points to have Anosov geodesic flow.

The next result is a set of variations on the theme that in a manifold with no focal points most Jacobi fields $Y$ with $Y(0) = 0$ have $\|Y(t)\|$ increasing for $t > 0$.

4.13 PROPOSITION

The following three properties of a Riemannian manifold $M$ are equivalent.

(i) $M$ has no focal points.

(ii) Let $Y$ be a non-trivial, initially vanishing, orthogonal Jacobi field along a geodesic $\gamma$ in $M$. Then

$\langle Y,Y\rangle'(t) > 0$ for $t > 0$. (We shall see in (5.11) that $\|Y(t)\| \to \infty$ as $t \to \infty$.)

(iii) Along any geodesic $\gamma$ in M, $A(t)$ is non-singular and $A'A^{-1}(t) > 0$ for all $t > 0$.

Any of the above implies that along any geodesic $\gamma$ in M

(iv) $A^*A(r) < A^*A(s)$ when $0 < r < s$;

(v) $\|A(t)\|$ and $((A(t)))$ are both strictly increasing for $t > 0$.

This result is contained in Theorem 11 of [20].

Proof.

(i) $\Rightarrow$ (ii). Since $\langle Y,Y\rangle'(0) = 0$ and $\langle Y,Y\rangle'' = 2\langle Y',Y'\rangle(0) > 0$, we have $\langle Y,Y\rangle'(t) > 0$ for small positive t. Now suppose (ii) is false. Then there is $s > 0$ with $\langle Y,Y\rangle'(s) = 0$ and $\langle Y,Y\rangle'(t) > 0$ for $0 < t < s$. Clearly $Y(0)$ is a focal point of $Y(s)$.

(ii) $\Rightarrow$ (iii) The Jacobi fields belonging to A are the initially vanishing orthogonal Jacobi fields. Thus (ii) says that for any non-zero $v \in \dot{\gamma}(0)^{\perp}$ and any $t > 0$, we have $0 < \langle Av,Av\rangle'(t) = 2\langle A'(t)v,A(t)v\rangle$. It is clear from this that when $t > 0$ we have $A(t)$ non-singular and $\langle A'A^{-1}(t)w,w\rangle > 0$ for all non-zero $w \in \dot{\gamma}(t)^{\perp}$.

(iii) $\Rightarrow$ (i). Suppose (i) is false. Then we can find a geodesic $\gamma$ and $b > 0$ such that $\gamma(0)$ is a focal point of $\gamma(b)$ along $\gamma$. There is a Jacobi field Y with $Y(0) = 0$, $Y(b) \neq 0$ and $\langle Y,Y\rangle'(b) = 0$. It is clear from (2.7) that Y is orthogonal. We see that Y belongs to A and so $Y'(b) = A'A^{-1}(b)\ (Y(b))$. Hence

$$\langle A'A^{-1}(b)(Y(b)), Y(b)\rangle = \tfrac{1}{2}\langle Y,Y\rangle'(b) = 0,$$

contrary to (iii).

(iii) $\Rightarrow$ (iv). It will suffice to show that for any non-zero $v \in \dot{\gamma}(0)^{\perp}$, the function $f_v(t) = \langle A^*A(t)v,v\rangle$ is strictly increasing for $t > 0$. Since $f_v = \langle A(t)v, A(t)v\rangle$,

$$f_v'(t) = 2\langle A'(t)v, A(t)v\rangle,$$

$$= 2\langle A'A^{-1}(t)w,w\rangle,$$

where $w = A(t)v \in \dot{\gamma}(t)^{\perp}$. We see from (iii) that $f_v'(t) > 0$ when $t > 0$.

(iv) $\Rightarrow$ (v). Since $A^*A(t)$ is symmetric and positive definite, we know from §3A that $\|A^*A(t)\| = \lambda^+(A^*A(t))$. By (iv), this is strictly increasing when $t > 0$. Since $\langle A^*A(t)v,v\rangle = \langle A(t)v,A(t)v\rangle$ for all $v \in \dot{\gamma}(0)^{\perp}$, we have $\|A^*A(t)\| = \|A(t)\|^2$. The argument for $((A(t)))$ is similar. $\square$

Geometrically, the no focal points property says that distance functions and geodesic balls in the universal cover H are convex. We call a function $f: \mathbb{R} \to \mathbb{R}$ *convex* if $f(\lambda a + (1-\lambda)b) \leq \lambda f(a) + (1-\lambda)f(b)$ for any $\lambda \in (0,1)$ and $a,b \in \mathbb{R}$, and *strictly convex* if this inequality is always strict when $a \neq b$. A real valued function f on a Riemannian manifold H is (strictly) convex if $f \circ \gamma$ is (strictly) convex for every geodesic $\gamma$. If f is $C^2$, it is convex if and only if its Hessian $\nabla^2 f$ is positive semi definite. A subset C of H is *convex* if any geodesic segment joining two points of C lies in C. Sublevel sets of convex functions are convex. If C is convex, $p \in C$ is *extreme* if p does not lie in the interior of a geodesic segment contained in C. A convex set C will be called *strictly convex* if it is the closure of an open set and every point of $\partial C$ is extreme. Clearly C is strictly convex if $\partial C$ is a smoothly embedded hypersurface whose second fundamental tensor relative to the outward unit normal is everywhere positive definite.

4.14 PROPOSITION

Assume M has no conjugate points. Then the following properties of its universal cover H are equivalent.

(i) H has no focal points.

(ii) For any given $p_0 \in H$ the distance function $f = d(p_0, \cdot)$ is convex, and is strictly convex along any geodesic which does not pass through $p_0$.

(iii) For any $p_0 \in H$ and any $r > 0$, the closed geodesic ball $\bar{B}(p_0, r)$ is strictly convex.

(iv) For any $p_0 \in H$ and any $r > 0$, $\bar{B}(p_0, r)$ is convex.

Compare Lemma 1 of [18] and Theorem 11 of [20].

Proof.

(i) $\Rightarrow$ (ii). Firstly, f is convex along geodesics through $p_0$. For if $\gamma$ is a geodesic with $\gamma(0) = p_0$, we see from (4.8iv) that $f(\gamma(t)) = |t|$.

We know from (4.10) that f is smooth at any point $p \neq p_o$. If $\gamma$ is a geodesic and $\gamma(t) = p$, then $(f\circ\gamma)''(t) = \nabla^2 f(p)(\dot\gamma(t), \dot\gamma(t))$. We are led to study $\nabla^2 f(p)$. Define the vector field on $H\setminus\{p_o\}$, $W = -V(\cdot, p_o)$. It will suffice to prove the following claim: for all $w \in T_pH$,

$$\nabla^2 f(p)(w,w) \geq 0,$$

with equality only if $w = W(p)$.

By (4.10i), grad $f = W$ on $H\setminus\{p_o\}$. Hence

$$\nabla^2 f(p)(w,w) = \langle \nabla_w W, w\rangle$$

for any $w \in T_pH$. By (4.10ii), $\nabla_{W(p)}W = 0$. Also on $W(p)^\perp$, $w \to \nabla_w W$ is the second fundamental tensor, II, of the sphere $S(p_o, f(p))$ relative to $W(p)$. The claim will be proved if we show that II is positive definite. By (3.7), $II = A'A^{-1}(f(p))$, where A is the Lagrange map along $\gamma_{p_op}$ with $A(0) = 0$ and $A'(0) = I$. It now follows from (4.13iii) that II is positive definite.

(ii) $\Rightarrow$ (iii). $\bar B(p_o,r)$ is convex because it is a sublevel set of the convex function f. It is clear that $\bar B(p_o,r)$ is strictly convex, unless there is a geodesic $\gamma$ and $a < b < c$ with $f(\gamma(t)) \leq r$ for $a < t < c$ and $f(\gamma(b)) = r$. Since $f\circ\gamma$ is convex, this is impossible unless $f(\gamma(t)) = r$

for $a < t < c$ and $f(\gamma(t)) > r$ for all $t$. But this means that $f \circ \gamma$ is not strictly convex and $\gamma$ does not go through $p_o$, which is impossible.

(iii) $\Rightarrow$ (iv). Trivial.

(iv) $\Rightarrow$ (i). By (4.13iii), it will suffice to show that along any geodesic $\gamma$ we have $A(t)$ non-singular and $A'A^{-1}(t) > 0$ for any $t > 0$. Since H has no conjugate points, $A_a(r)$ is non-singular whenever $r > a$. It is clear from (3.7) that if $A_a'A_a^{-1}(r)$ has a negative eigenvalue for some $r > a$, then $\bar{B}(\gamma(a), r-a)$ is not convex. Hence $A_a'A_a^{-1}(r) > 0$ for all $r > a$.

Now if $t > 0$, we know that $A_{-1}(r)$ is non-singular and $A_{-1}'A_{-1}^{-1}(r) > 0$ for $0 < r < 2t$. We see from (3.12) (with $L = A_{-1}$) that $A'A^{-1}(r) > A_{-1}'A_{-1}^{-1}(r)$ for $0 < r < 2t$. Taking $r = t$ gives us $A'A^{-1}(t) > 0$. $\square$

Like having no conjugate points, having no focal points is a closed condition on Riemannian metrics.

4.15 PROPOSITION

Let M be a manifold. The set of Riemannian metrics on M with focal points is open in the Whitney $C^1$ topology on $\mathcal{R}(M)$.

Proof

Since every metric with conjugate points has focal points (4.12) and the set of metrics with conjugate points is open (4.11), all we need to prove is that having focal points is an open condition on metrics with no conjugate points. This follows from (4.14iv). The argument is similar to the proof of (4.11) and will not be given. □

## D. GEOMETRY WITH NO FOCAL POINTS

Let H be a simply connected manifold with no focal points. We prove some simple results which follow from the convexity of distance functions on H.

### 4.16 LEMMA

Suppose $p \in H$ and $\gamma$ is a geodesic that does not pass through p. Then the function $\theta(t) = \measuredangle_{\gamma(t)}(p, \dot{\gamma}(t))$ is strictly increasing.

**Proof.**

Write $g(t) = d(p,\gamma(t))$. By (4.10i),

$$g'(t) = \langle \dot{\gamma}(t), -V(p,\gamma(t)) \rangle = -\cos\theta(t).$$

Since g is strictly convex (by 4.14ii), it follows that $\theta$ is strictly increasing. □

### 4.17 LEMMA

Suppose $p \in H$ and $\gamma$ is a geodesic that does not pass through p. Then there is a unique perpendicular from p to $\gamma$. Its foot is the unique point on $\gamma$ closest to p.

Proof.

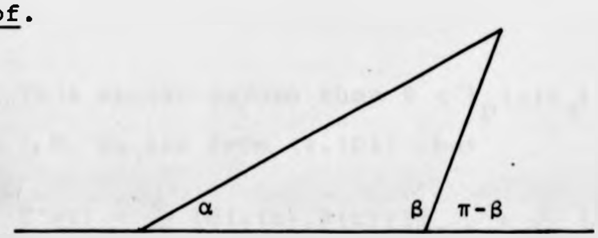
Again write $g(t) = d(p,\gamma(t))$. It is clear from (4.10i) that $\gamma(t)$ is the foot of a perpendicular from p to $\gamma$ if and only if $g'(t) = 0$. It is clear that g has a minimum, which is its only critical point, since g is strictly convex (by 4.14ii). □

In fact the above property is equivalent to the no focal points property [18, Proposition 2; 20 Theorem 11].

4.18 LEMMA

The sum of any two angles in a geodesic triangle is $< \pi$ (unless all three vertices lie on a single geodesic).

Proof.




$$\alpha < \pi-\beta \text{ by } (4.16). \qquad \square$$

4.19 PROPOSITION

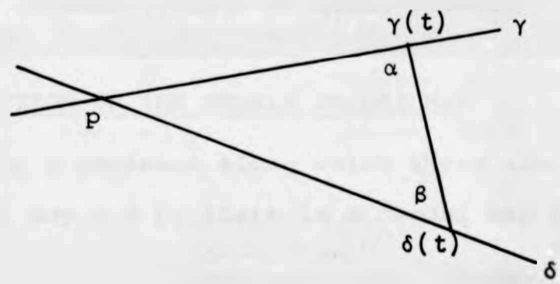
Let $\gamma$ and $\delta$ be geodesics that intersect at $p = \gamma(t_1) = \delta(t_2)$. If $\dot{\gamma}(t_1) = \dot{\delta}(t_2)$, then $d(\gamma(t),\delta(t))$ is constant. Otherwise it is strictly increasing when $t > M = \max(t_1,t_2)$ and strictly decreasing when $t < m = \min(t_1,t_2)$.

Compare Proposition 2 of [37].

<u>Proof</u>.

Write $f(t) = d(\gamma(t),\delta(t))$.

If $\dot{\gamma}(t_1) = \dot{\delta}(t_2)$, then $f(t) \equiv |t_1-t_2|$. If $\dot{\gamma}(t_1) = -\dot{\delta}(t_2)$, then $f(t) = |(t-t_1) + (t-t_2)|$, which is strictly increasing if $t > M$ and strictly decreasing if $t < m$.




Thus we can assume that $0 < \measuredangle_p(\dot{\gamma}(t_1), \dot{\delta}(t_2)) < \pi$. If $t > M$, we see from (4.10i) that

$$f'(t) = \frac{d}{ds}\{d(\gamma(s),\delta(t))\}|_{s=t} + \frac{d}{ds}\{d(\gamma(t),\delta(s))\}|_{s=t}$$

$$= \cos\alpha + \cos\beta,$$

where $\alpha$ and $\beta$ are the angles in the picture. By (4.18), $0 < \alpha + \beta < \pi$. Hence $0 < \cos\alpha + \cos\beta = f'(t)$.

A similar argument shows that $f'(t) < 0$ when $t < m$. □

## §5. THE STABLE JACOBI MAP

Along any geodesic in a manifold with no conjugate points is a Jacobi map - the stable Jacobi map - whose properties are fundamental in the study of the geodesic flow. It is constructed as a certain limiting solution of the Jacobi map equation. The properties that are wanted in the study of the geodesic flow can all be proved if the manifold is assumed to have no focal points.

### A. CONSTRUCTION OF THE STABLE JACOBI MAP

Let $\gamma$ be a geodesic along which there are no conjugate points. For any $r \neq 0$, there is a Jacobi map $D_r$ along $\gamma$ with

$$D_r(0) = I \text{ and } D_r(r) = 0.$$

Indeed

$$D_r(t) = A_r(t) \circ A_r(0)^{-1},$$

where $A_r$ is as defined in §3C. It follows from this and (4.1) that $D_r(t)$ is non-singular unless $t = r$. Since it is obvious that $W(D_r, D_r)(r) = 0$, we see that $D_r$ is a Lagrange map. It follows, since $D_r'(0) = D_r'D_r^{-1}(0)$, that $D_r'(0)$ is symmetric.

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5.1 <u>REMARK</u>

Clearly a Jacobi field belongs to $D_r$ if and only if it belongs to $A_r$. It is obvious from §3C that $A_r$ is related to the family of spheres with centre $\gamma(r)$. It follows that $D_r$ is also related to these spheres. In particular, $D_r'(0) = D_r'D_r^{-1}(0)$ is the second fundamental tensor of $S(\gamma(r), |r|)$ relative to $\dot{\gamma}(0)$.

Now we show that $D_r$ converges as $r \to \infty$, following the original argument of Green [28]. Since $D_r(0) = I$ for any $r$, all we need to show is that $D_r'(0)$ converges (see 3.1ii). Since $D_r'(0)$ is always symmetric, we can state the following.

5.2 <u>LEMMA</u> [28]

$D_r'(0) < D_s'(0) < D_{-1}'(0)$ when $0 < r < s$.

<u>Proof</u>

We give a new proof.

Since $D_s(t)$ is non-singular for $-1 < t < r$, (3.12) with $L = D_s$ gives

$$D_{-1}'D_{-1}^{-1}(t) > D_s'D_s^{-1}(t) > D_r'D_r^{-1}(t)$$

for $-1 < t < r$. Take $t = 0$ and recall that $D_a(0) = I$ for any $a \neq 0$. □

<u>Remarks</u>

(i) Green's proof in [28] was based on (5.7) below.

(ii) Lemma 4.6 is essentially the same result, since $D_r'D_r^{-1}(t) = A_r'A_r^{-1}(t)$ for $t \neq r$.

The convergence of $D_r'(0)$ as $r \to \infty$ is immediate from the previous lemma and the following.

5.3 <u>LEMMA</u>

Let V be an inner product space and S(V) the space of symmetric endomorphisms of V. Suppose $L_n$ is a sequence in S(V) such that $L_m < L_n$ when $m < n$. Suppose there is $B \in S(V)$ such that $L_n < B$ for all n. Then there is $L \in S(V)$ such that $L_n \to L$.

<u>Proof</u>

For any $v \in V$, $\langle L_n v, v\rangle$ is a non-decreasing sequence bounded above by $\langle Bv,v\rangle$ and so is convergent as $n \to \infty$. Since $\langle L_n v,w\rangle = \frac{1}{2}\{\langle L_n(v+w), (v+w)\rangle - \langle L_n v,v\rangle - \langle L_n w,w\rangle\}$, we see that $\langle L_n v,w\rangle$ converges as $n \to \infty$ for any $v,w \in V$. Define L by $\langle Lv,w\rangle = \lim_{n\to\infty} \langle L_n v,w\rangle$. □

In a similar way we can show that $D_r$ converges as $r \to -\infty$.

Now suppose that M is a manifold with no conjugate points. We are justified in making the following definitions in which we use the notation of §3F.

5.4 DEFINITION

For $v \in SM$, let $D_r(v,t)$ be the Jacobi map along $\gamma_v$ with

$$D_r(v,0) = I \text{ and } D_r(v,r) = 0.$$

The *stable* and *unstable Jacobi maps* along $\gamma_v$ are the Jacobi maps $D^s(v,t)$ and $D^u(v,t)$ respectively with

$$D^s(v,0) = I = D^u(v,0),$$

$$D^{s\prime}(v,0) = \lim_{r\to\infty} D_r'(v,0)$$

and

$$D^{u\prime}(v,0) = \lim_{r\to-\infty} D_r'(v,0).$$

The *stable* and *unstable Jacobi fields* are those belonging to the stable and unstable Jacobi maps respectively.

When it is clear which geodesic is under consideration, we shall suppress the dependence on v in the notation.

Examples

Along any geodesic in a manifold with constant negative curvature $-k^2$,

$$D_r(t) = \left\{\frac{e^{kr}}{e^{kr}-e^{-kr}}\, e^{-kt} - \frac{e^{-kr}}{e^{kr}-e^{-kr}}\, e^{kt}\right\} \pi_0^t,$$

and so $D^s(t) = e^{-kt}\,\pi_0^t$ and $D^u(t) = e^{kt}\pi_0^t$.

Along a geodesic in a flat manifold,

$$D_r(t) = (1 - \frac{t}{r})\ \pi_0^t,$$

and so $D^s(t) = \pi_0^t = D^u(t)$.

We make some simple observations.

5.5 LEMMA

Let $\gamma$ be a geodesic in a manifold with no conjugate points.

(i) $D^s(\dot\gamma(0),t)$ and $D^u(\dot\gamma(0),t)$ are Lagrange tensors.

(ii) $D^s(\dot\gamma(0),t) = D^u\ (-\ \dot\gamma(0),\ -t)$.

(iii) For any s and t,

$$D^s(\dot\gamma(0),t) = D^s(\dot\gamma(s),t-s) \circ D^s(\dot\gamma(0),s)$$

and

$$D^u(\dot\gamma(0),t) = D^u(\dot\gamma(s),t-s) \circ D^u(\dot\gamma(0),s).$$

<u>Proof</u>

(i) This follows from $W(D_r,D_r)\ (\dot\gamma(0),t) \equiv 0$ by letting $r \to \infty$ and $r \to -\infty$

(ii) Obvious.

(iii) This will follow by letting $r \to \infty$ and $r \to -\infty$ if we show that

$$D_r(\dot\gamma(0),t) = D_{r-s}(\dot\gamma(s),t-s) \circ D_r(\dot\gamma(0),s)$$

whenever $r \neq 0, s$. But (for fixed r and s) both sides are Jacobi maps, and they are equal when $t = r$ and $t = s$. Consider the difference of the two sides. It is a Jacobi map and any Jacobi field belonging to it vanishes twice and hence vanishes identically, since there are no conjugate points. It follows that the two sides are equal. □

The Lagrange maps $D^s$ and $D^u$ and their related hypersurfaces - the horospheres which we will consider in §6 - are basic to the study of hyperbolic properties of the geodesic flow $\Phi_t$. If $\Phi_t$ is hyperbolic along the orbit $\{\dot{\gamma}(t)\}$, then the stable and unstable subspaces of $T_{\dot{\gamma}(0)}SM$ are those corresponding to the stable and unstable Jacobi fields respectively. The next result suggests why this should be the case.

5.6 PROPOSITION [22, Proposition 3']

Let $\gamma$ be a geodesic with no conjugate points and L a Lagrange map with L(0) non-singular. Then L(t) is non-singular for all t if and only if

$$D^{s\prime}(0) < L'L^{-1}(0) < D^{u\prime}(0).$$

In particular, $D^s$ and $D^u$ are everywhere non-singular. Furthermore, for any L which has L(t) non-singular for all t,

$$D^{s\prime}D^{s^{-1}}(t) < L'L^{-1}(t) < D^{u\prime}D^{u^{-1}}(t)$$

for all t.

Thus the stable Jacobi map contracts and the unstable Jacobi map expands faster than any other everywhere non-singular Lagrange map.

Proof

Suppose $L(0)$ is non-singular and

$$D^{s'}(0) < L'L^{-1}(0) < D^{u'}(0). \qquad (*)$$

It is clear from its construction that $D^{s'}(0) > D_r'(0)$ for any $r > 0$. Similarly, $D^{u'}(0) < D_{-r}'(0)$ for any $r > 0$. It follows that for any $r > 0$

$$D_r'D_r^{-1}(0) < L'L^{-1}(0) < D_{-r}'D_{-r}^{-1}(0).$$

Lemma 3.11 with $U = D_r'D_r^{-1}$ says that $L(t)$ is non-singular for $r > t > 0$; with $U = D_{-r}'D_{-r}^{-1}$ it says that $L(t)$ is non-singular for $-r < t < 0$. Thus $L$ is everywhere non-singular, and so are $D^s$ and $D^u$ since they also satisfy (*). It now follows from (*) and (3.10) that

$$D^{s'}D^{s^{-1}}(t) < L'L^{-1}(t) < D^{u'}D^{u^{-1}}(t)$$

for all $t$.

Conversely, if $L$ is everywhere non-singular (3.12) says that if $r > 0$,

$$D_{-r}'D_{-r}^{-1}(t) = A_{-r}'A_{-r}^{-1}(t) > L'L^{-1}(t) > A_r'A_r^{-1}(t) = D_r'D_r^{-1}(t)$$

for $-r < t < r$. Let $r \to \infty$. □

We can apply (3.16) to express $D^s$ in terms of A.

5.7 LEMMA [28]

Along any geodesic which has no conjugate points, we have for $t > 0$ and $r \neq 0$,

(i) $D_r(t) = A(t) \int_t^r (A^*A)^{-1}(x)dx;$

(ii) $D^s(t) = A(t) \int_t^\infty (A^*A)^{-1}(x)dx.$

Proof [21, p.240]

(i) Since $D_r(r) = 0$ and $W(D_r, A)(0) = -I$, (3.16) gives

$$D_r(t) = -A(t) \int_r^t (A^*A)^{-1}(x)dx.$$

(ii) Let $r \to \infty$ in (i). □

## B. THE STABLE JACOBI MAP IN MANIFOLDS WITH NO FOCAL POINTS

Our first observation is an easy consequence of (4.13).

5.8 PROPOSITION

Along any geodesic in a manifold with no focal points we have the following.

(i) $D^{s*}D^s(t_1) > D^{s*}D^s(t_2)$ and $D^{u*}D^u(t_1) < D^{u*}D^u(t_2)$

whenever $t_1 < t_2$.

(ii) $D^{s'}D^{s^{-1}}(t) < 0 < D^{u'}D^{u^{-1}}(t)$ for all t.

(iii) The length of a stable (unstable) Jacobi field is non-increasing (non-decreasing).

(iv) $\|D^s(t)\|$ and $((D^s(t)))$ are non-increasing;

$\|D^u(t)\|$ and $((D^u(t)))$ are non-decreasing.

Compare Theorem 11 of [20].

<u>Proof</u>.

We prove the results for $D^u$; those for $D^s$ then follow by (5.5ii).

(i) Since $D_r(t) = A_r(t) \circ A_r^{-1}(0)$, we have $D_r^*D_r(t) = A_r^{-1*}(0) \circ A_r^*A_r(t) \circ A_r^{-1}(0)$. It follows from (4.13iv) that

$$A_r^*A_r(t_1) < A_r^*A_r(t_2) \text{ if } r \leq t_1 < t_2.$$

Hence $D_r^*D_r(t_1) < D_r^*D_r(t_2)$ if $r < t_1 < t_2$. Now let $r \to -\infty$.

(ii) Similarly, $D_r'D_r^{-1}(t) = A_r'A_r^{-1}(t) > 0$ when $t > r$ by (4.13iii). As $r \to -\infty$, $D_r'D_r^{-1} \to D^{u'}D^{u-1}$.

(iii) and (iv) follow easily from (i). □

It will be essential in our investigation of horospheres in the next section that $D^s(v,t)$ and $D^{s'}(v,t)$ depend continuously on v and t. Since $D^s(v,0) = I$ for any v, this will be the case if $D^{s'}(v,0)$ depends continuously on v.

This in turn will follow *if the convergence of* $D_r'(v,0)$ *to* $D^{s'}(v,0)$ *as* $r \to \infty$ *is uniform for all* $v$. It is still an open question whether this can be proved for all manifolds with no conjugate points. It was proved for manifolds with non-positive curvature by Heintze and Im Hof [31] and by Eberlein (unpublished, cited in [31]). Eschenburg [20, 21] has given a proof for a class of manifolds which includes those with no focal points and those with Anosov geodesic flow (see (5.10) below).

Our aim is to study manifolds with no focal points, so we now present Eschenburg's proof in that context. Consider a geodesic $\gamma$ in a manifold with no focal points. First we show that along $\gamma$

$$D^{s'}(0) - D_r'(0) = A^{-1}D^s(r) \qquad (*)$$

for any $r \neq 0$. We have

$$D^s(t) - D_r(t) = A(t)[D^{s'}(0) - D_r'(0)],$$

since both sides are Jacobi maps that vanish and have the same derivative at $t = 0$. Taking $t = r$ gives us (*), since $D_r(r) = 0$.

The next step is to apply (3.16). Since $D^{s^{-1}}A(0) = 0$ and $W(A,D^s)(0) = I$, we see that

$$A(r) = D^s(r)\int_0^r (D^{s*}D^s)^{-1}(x)dx.$$

Thus

$$D^{s^{-1}}A(r) = \int_0^r (D^{s*}D^s)^{-1}(x)dx,$$

and so

$$\|A^{-1}D^s(r)\|^{-1} = ((D^{s^{-1}}A(r)))$$

$$> \int_0^r (((D^{s*}D^s)^{-1}(x)))dx$$

$$= \int_0^r \|D^{s*}D^s(x)\|^{-1}dx.$$

Since $D^{s*}D^s(t)$ is symmetric and $\langle D^{s*}D^s(t)v,v\rangle = \langle D^s(t)v,D^s(t)v\rangle$ for all $v \in \dot{\gamma}(0)^\perp$, we have $\|D^{s*}D^s(x)\| = \|D^s(x)\|^2$. Hence

$$\|A^{-1}D^s(r)\|^{-1} > \int_0^r \|D^s(x)\|^{-2}dx$$

$$> r\,\|D^s(0)\|^{-2}$$

$$= r,$$

since $\|D^s(t)\|$ is non-increasing (by 5.8iv). Thus

$$\|D^{s'}(0) - D_r'(0)\| = \|A^{-1}D^s(r)\| < 1/r. \qquad (**)$$

We have proved

5.9 <u>PROPOSITION</u>

Let M be a Riemannian manifold with no focal points. Then $D_r'(v,0) \to D^{s'}(v,0)$ as $r \to \infty$ uniformly for all $v \in SM$.

The maps $(v,t) \to D^s(v,t)$ and $(v,t) \to D^{s'}(v,t)$ are continuous. So are $(v,t) \to D^u(v,t)$ and $(v,t) \to D^{u'}(v,t)$; this follows from the previous sentence since $D^u(v,t) = D^s(-v,-t)$ by (5.5ii).

5.10 REMARK

Eschenburg considered manifolds with *bounded asymptote:* a manifold with no conjugate points has bounded asymptote if there is a number $\rho \geq 1$ such that $\|D^s(v,t)\| \leq \rho$ for any $v \in SM$ and any $t \geq 0$. In place of (**) he obtained $\|D^{s'}(0) - D'(0)\| \leq \rho^2/r$. For manifolds with no focal points it follows from (5.8iv) that we can take $\rho = 1$.

We next consider the growth of the Jacobi map $A(v,t)$ as $t \to \infty$. We saw in (4.13) that in manifolds with no focal points $((A(v,t)))$ is increasing for $t > 0$.

5.11 PROPOSITION

Along any geodesic $\gamma$ in a manifold with no focal points, $((A(t))) \geq \frac{1}{2} t$ for any $t \geq 0$.

Remarks

This estimate was proved for surfaces by Berger [7]. Eschenburg and O'Sullivan [22, Proposition 4] have proved an estimate of the form $((A(v,t))) \geq \text{constant}.\sqrt{t}$ for all $v \in SM$. Their proof requires a lower bound on the curvature but holds for any manifold with bounded asymptote. Green [27, Lemma 2] tried to prove essentially that $((A(t))) \to \infty$ as $t \to \infty$ along every geodesic in a manifold with no conjugate points and curvature bounded from below. His proof was found to be incomplete by Eberlein [14, p. 168]. Goto [25, Theorem 1] gave the first proof that $((A(t))) \to \infty$ along all geodesics in manifolds with no focal points. The argument below is a simple modification of her proof.

Proof of (5.11)

The proof is based on the inequality (**) that we proved above:

$$||A^{-1}D^{s}(t)|| < \frac{1}{t},$$

for any $t > 0$. It follows from this and (5.7ii) that, for any $t > 0$,

$$|| \int_t^{\infty} (A^*A)^{-1}(x)dx|| < \frac{1}{t}.$$

Hence for any unit vector $u \in \dot{\gamma}(0)^{\perp}$ and any $t > 0$,

$$\int_t^{\infty} <(A^*A)^{-1}(x)u,u>du < \frac{1}{t}.$$

We need to know that the integrand is non-negative and non-increasing for $x > 0$. It is non-negative, since $<(A^*A)^{-1}(x)u,u> = <A^{-1*}(x)u,\ A^{-1*}(x)u>$. To show that it is non-increasing, write $F(x) = A^*A(x)$. It will suffice to show that $F^{-1'}(x) < 0$ when $x > 0$. Since $F(x)$ is always symmetric,

$F'(x)$ is symmetric. It is clear from (4.13iv) that $F'(x) > 0$ for $x > 0$. Hence $(F^{-1})'(x)$ is symmetric and $(F^{-1})'(x) = -F^{-1}F'F^{-1}(x) < 0$ for $x > 0$.

Now we can see that

$$\frac{4}{t^2} > \langle A^*A)^{-1}(t)u,u\rangle. \qquad (***)$$

For, since the integrand is non-negative and non-increasing,

$$\begin{aligned} \frac{2}{t} &> \int_{t/2}^{\infty} \langle (A^*A)^{-1}(x)u,u\rangle du \\ &> \int_{t/2}^{t} \langle (A^*A)^{-1}(x)u,u\rangle du \\ &> \frac{t}{2} \langle (A^*A)^{-1}(t)u,u\rangle. \end{aligned}$$

Since u can be any unit vector in $\dot{\gamma}(0)^{\perp}$, it follows from (***) that

$$\frac{4}{t^2} > \|(A^*A)^{-1}(t)\| = ((A^*A(t)))^{-1}.$$

Since $A^*A(t)$ is symmetric and $\langle A^*A(t)v,v\rangle = \langle A(t)v,A(t)v\rangle$ for any $v \in \dot{\gamma}(0)^{\perp}$, we have $((A^*A(t))) = ((A(t)))^2$. Hence

$$((A(t))) > \frac{2}{t}. \qquad \square$$

5.12 <u>COROLLARY</u>

Let Y be an orthogonal Jacobi field along a geodesic in a manifold with no focal points.

(i) $Y$ is stable if and only if $\|Y(t)\|$ is bounded as $t \to \infty$.

(ii) $Y$ is unstable if and only if $\|(Y(t))\|$ is bounded as $t \to -\infty$.

(iii) The following are equivalent.

(a) $\|Y(t)\|$ is bounded for all $t$.

(b) $Y$ is both stable and unstable.

(c) $D^{s'}(0)(Y(0)) = D^{u'}(0)(Y(0))$.

(d) $Y$ is parallel (i.e. $Y' = 0$).

<u>Proof</u> [17, Proposition 2.12 and pp.458,459]

(i) We can write $Y = Y_1 + Y_2$ where $Y_1$ is the stable Jacobi field with $Y_1(0) = Y(0)$ and $Y_2$ is an orthogonal Jacobi field with $Y_2(0) = 0$. By (5.8iv), $\|Y_1(t)\|$ is bounded as $t \to \infty$; and by (5.11) $\|Y_2(t)\| \to \infty$ as $t \to \infty$, unless $Y_2 = 0$.

(ii) Follows from (i) by (5.5ii).

(iii) (a) $\Longleftrightarrow$ (b): is clear from (i) and (ii). (b) $\Longleftrightarrow$ (c) and (d) $\Rightarrow$ (a): obvious. (b) $\Rightarrow$ (d): If $Y$ is both stable and unstable,

$$D^{s'}D^{s^{-1}}(t)(Y(t)) = Y'(t) = D^{u'}D^{u^{-1}}(t)(Y(t))$$

for all $t$. Since (by 5.8ii)

$$D^{s'}D^{s^{-1}}(t) < 0 < D^{u'}D^{u^{-1}}(t),$$

this is possible only if $Y'(t) \equiv 0$. ▫

(i) $Y$ is stable if and only if $\|Y(t)\|$ is bounded as $t \to \infty$.

(ii) $Y$ is unstable if and only if $\|(Y(t))\|$ is bounded as $t \to -\infty$.

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$$D^{s'}D^{s^{-1}}(t)(Y(t)) = Y'(t) = D^{u'}D^{u^{-1}}(t)(Y(t))$$

for all $t$. Since (by 5.8ii)

$$D^{s'}D^{s^{-1}}(t) < 0 < D^{u'}D^{u^{-1}}(t),$$

this is possible only if $Y'(t) \equiv 0$. $\square$

Pesin [39, Proposition 4.11(5); 40, Proposition 3(5); 41, Theorem 2.2(7)] asserts that (i) and (ii) hold in any manifold with no conjugate points whose curvature is bounded from below. This has not been proved so far. For a detailed discussion see Remark II of Section II in the preprint of [24].

## C. DIVERGENCE OF GEODESICS IN MANIFOLDS WITH NO FOCAL POINTS

Throughout this subsection, H will be a simply connected manifold with no focal points. We saw in (4.19) that the distance between two intersecting geodesics in H increases as we move away from the point where they intersect. Now we see that it becomes unbounded. The proof is based on the corresponding infinitesimal result (5.11).

### 5.13 PROPOSITION

Let $\gamma$ and $\delta$ be geodesics in H with $\gamma(0) = p = \delta(0)$. For any $R > 0$,

$$d(\gamma(t_1), \delta(t_2)) \geq \frac{R}{4} \measuredangle_p (\dot{\gamma}(0), \dot{\delta}(0))$$

whenever $t_1, t_2 \geq R$.

Together with (4.19) this gives a slightly sharper version of [37, Proposition 2] and [25, Theorem 2]. The argument used is similar to the proof of [37, Lemma 1].

Proof

Write $\theta = \measuredangle_p (\dot{\gamma}(0), \dot{\delta}(0))$. Suppose $\sigma:[0,1] \to H$ is a smooth curve joining $\gamma(t_1)$ to $\delta(t_2)$. We show that $\ell(\sigma) \geq \frac{1}{4} R\theta$.

If $\sigma$ enters the ball $B(p,R/2)$, we see that $\ell(\sigma) \geq R > \frac{1}{4} R\pi \geqslant \frac{1}{4} R\theta$.

Thus we can, and do, assume that $r(s) = d(p,\sigma(s)) \geq \frac{1}{2} R$ for all $s \in [0,1]$. Note that $r$ is smooth, since $\sigma$ does not pass through $p$. Consider the geodesic variation,

$$\alpha(s,t) = \gamma_{p\sigma(s)}(t).$$

The vector fields $\frac{\partial\alpha}{\partial t}$ and $\frac{\partial\alpha}{\partial s}$ are mutually orthogonal, since they are respectively orthogonal and tangential to the spheres with centre $p$. Hence

$$\begin{aligned} ||\dot{\sigma}(x)|| &= ||\frac{\partial\alpha}{\partial s}(x,r(x)) + r'(x)\frac{\partial\alpha}{\partial t}(x,r(x))|| \\ &\geq ||\frac{\partial\alpha}{\partial s}(x,r(x))|| \end{aligned}$$

for any $x \in [0,1]$. Thus

$$\ell(\sigma) \geq \int_0^1 ||Y_x(r(x))||\, dx,$$

where $Y_x(t) = \frac{\partial\alpha}{\partial s}(x,t)$.

Now $Y_x$ is an initially vanishing orthogonal Jacobi field along $\delta_x = \gamma_{p\sigma(x)}$. Since $r(x) \geq \frac{1}{2} R$, it follows from (4.13ii) that

$$\|Y_x(r(x))\| \geq \|Y_x(R/2)\| .$$

Clearly $Y_x$ belongs to the Jacobi map $A_x(t) = A(\dot{\delta}_x(0),t)$ along $\delta_x$; it is easily checked that $Y_x(t) = A_x(t)(Y_x'(0))$. It follows using (5.11) that

$$\|Y_x(R/2)\| \geq ((A_x(R/2)))\ \|Y_x'(0)\|$$

$$\geq \frac{1}{4} R \|Y_x'(0)\| .$$

Hence

$$\ell(\sigma) \geq \frac{1}{4} R \int_0^1 \|Y_x'(0)\|\, dx$$

$$\geq \frac{1}{4} R\theta,$$

since $x \to Y_x'(0)$ is a curve in $S_pH$ joining $\dot{\gamma}(0)$ to $\dot{\delta}(0)$. □

5.14 COROLLARY

Suppose we have a curve in H with length L which does not approach within distance $R > 0$ of a point p. Then the angle subtended at p by its endpoints is at most $\frac{4L}{R}$.

Pesin [39, Proposition 4.4(3); 41, Proposition 1.3(3)] asserts that if $\gamma$ and $\delta$ are intersecting geodesics in a manifold with no conjugate points and curvature bounded from below, then $d(\gamma(t), \delta(t)) \to \infty$ as $t \to \infty$. This has yet to be proved.

## §6. THE SPHERE AT INFINITY AND HOROSPHERES

Throughout this section H will be a simply connected manifold with no focal points. No additional restrictions are placed on H. In parts A and B, H is compactified by the addition of a boundary sphere and a natural topology constructed on the resulting space. Any geodesic in H joins two points of the boundary sphere. The geodesics which end at a given boundary point have orthogonal hypersurfaces, known as horospheres. These are studied in part C. Part D considers geodesics which join the same two boundary points, while part E contains some results that will be needed in §7.

The theory described here was first developed for manifolds with non-positive curvature by Eberlein and O'Neill in [19]. Subsequent work - notably by Eschenburg [20, 21], O'Sullivan [37], Pesin [39] and Goto [25, 26] - has shown that Eberlein and O'Neill's theory still applies if H is assumed only to have no focal points. We give an exposition of this work; the only original result is (6.29).

### A. ASYMPTOTIC GEODESICS AND POINTS AT INFINITY

It is possible to construct a boundary sphere for H - the sphere of "points at infinity" - analogous to the boundary circle of the Poincaré disc. We begin by studying the families of geodesics that ought to have one or two common endpoints in the boundary sphere.

DEFINITION

Two geodesics in H, $\gamma$ and $\delta$, are *positively asymptotic* or simply *asymptotic* if $d(\gamma(t), \delta(t))$ is bounded for $t > 0$. They are *negatively asymptotic* if $d(\gamma(t),\delta(t))$ is bounded for $t < 0$, and *biasymptotic* if $d(\gamma(t), \delta(t))$ is bounded for all t.

6.1 REMARKS

(i) All three properties are equivalence relations.

(ii) Geodesics $\gamma$ and $\delta$ are negatively asymptotic if and only if $\bar{\gamma}$ and $\bar{\delta}$ are asymptotic.

(iii) If asymptotic geodesics have a point in common, then they are orientation preserving reparametrizations of one another; this is clear from (5.13).

6.2 EXAMPLES

(i) Geodesics in the Poincaré disc have a common endpoint on the boundary circle if and only if they are positively or negatively asymptotic.

(ii) [41, p.22]. Asymptotic geodesics need not approach one another. Parallel straight lines in the Euclidean plane (with the right choice of direction) are asymptotic (indeed biasymptotic).

The fundamental fact about asymptotic geodesics is:

6.3 PROPOSITION

Let p be a point and $\gamma$ a geodesic in H. (a) There is a unique geodesic $\delta$ with $\delta(0) = p$ that is asymptotic to $\gamma$. (b) We have

$$\dot{\delta}(0) = \lim_{s\to\infty} V(p,\gamma(s)).$$

(c) For every $L > 0$, this convergence is uniform for all p and $\gamma$ with $d(p,\gamma(0)) < L$. (d) The vector $\dot{\delta}(0)$ varies continuously as $(p, \dot{\gamma}(0))$ varies in $H \times SH$.

This is essentially the same result as Theorem 5.1 of [39], except that Pesin assumes a lower bound on the curvature. It is also implicit in the results of [26].

Proof

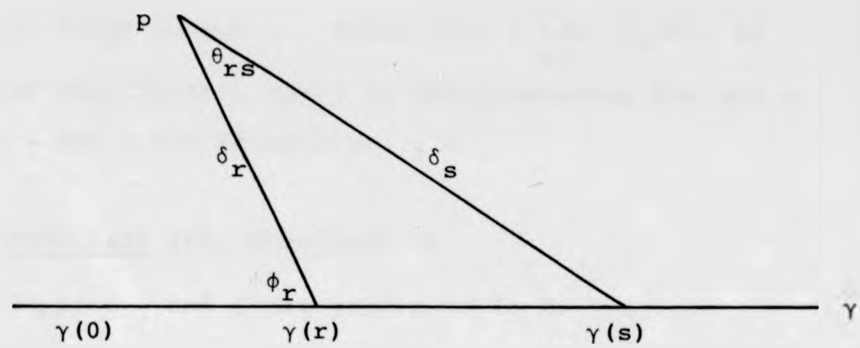
If p lies on $\gamma$, $V(p,\gamma(s))$ will be undefined for one - and only one - value of s. We assume in the following that r is large enough so $V(p,\gamma(s))$ is defined for all $s \geq r$.

It is clear from (5.13) that there can be at most one geodesic which starts at p and is asymptotic to $\gamma$.

Now we show that $V(p,\gamma(s))$ converges as $s \to \infty$ with the desired uniformity. We will show that the Cauchy property holds uniformly. If $0 < r < s$, write

$\theta_{rs} = \measuredangle_p(V(p,\gamma(r)), V(p,\gamma(s))) = \measuredangle_p(\gamma(r),\gamma(s))$ and

$\phi_r = \measuredangle_{\gamma(r)}(p,\gamma(0))$.




We want to show that $\theta_{rs} \to 0$ with the appropriate uniformity as $r,s \to \infty$. In the triangle $p, \gamma(r), \gamma(s)$ the sum of the angles at $p$ and $\gamma(r)$ is $\leq \pi$ (with equality if and only if $p$ lies on $\gamma$); this follows from (4.18). Since these angles are $\theta_{rs}$ and $\pi-\phi_r$ respectively, we see that $0 \leq \theta_{rs} \leq \phi_r$. By (5.14), $\phi_r \to 0$, and for every $L > 0$ this convergence is uniform for all $p$ and $\gamma$ with $d(p,\gamma(0)) \leq L$. Hence $\theta_{rs} \to 0$ with the desired uniformity as $r,s \to \infty$.

Let $\delta$ be the geodesic with $\dot{\delta}(0) = \lim\limits_{s\to\infty} V(p,\gamma(s))$. It follows easily from the uniformity of this limit that we proved above that $\dot{\delta}(0)$ depends continuously on $(p, \dot{\gamma}(0))$.

Finally we show that $\delta$ is asymptotic to $\gamma$. Write $\delta_s = \gamma_{p,\gamma(s)}$, so $\dot{\delta}_s(0) = V(p,\gamma(s))$. Since $\gamma(s) = \delta_s(d(p,\gamma(s)))$, (4.19) says that $d(\gamma(t),\delta_s(t))$ is non-increasing for all $t < m_s = \min(d(p,\gamma(s)),s)$. Clearly $m_s \to \infty$ as $s \to \infty$. Hence for any given $t_1$ and $t_2$ with $t_1 \leq t_2$,

$$d(\gamma(t_1),\delta_s(t_2)) \leq d(\gamma(t_2),\delta_s(t_2))$$

for all large enough s. Since $\dot{\delta}(0) = \lim_{s\to\infty} \dot{\delta}_s(0)$, it follows that $d(\gamma(t), \delta(t))$ is non-increasing for all t. Hence $\gamma$ and $\delta$ are asymptotic. □

6.4 <u>COROLLARY</u> [25, Corollary 1]

Suppose $\gamma$ and $\delta$ are geodesics in H.

(i) If they are asymptotic, $d(\gamma(t),\delta(t))$ is non-increasing.

(ii) If they are negatively asymptotic, $d(\gamma(t),\delta(t))$ is non-decreasing.

(iii) If they are biasymptotic, $d(\gamma(a+t),\delta(b+t))$ is independent of t for any a and b.

<u>Proof</u>

(i) This was proved at the end of the previous proof.

(ii) Apply (i) to $\bar{\gamma}$ and $\bar{\delta}$.

(iii) This follows from (i) and (ii), since the geodesics $t \to \gamma(a+t)$ and $t \to \delta(b+t)$ are both positively and negatively asymptotic. □

6.5 <u>DEFINITION</u>

A *point at infinity* is an equivalence class of positively asymptotic geodesics. Let $H(\infty)$ denote the set of all points at infinity, and write $\bar{H} = H \cup H(\infty)$. Pesin calls $H(\infty)$ "the absolute".

In part B we shall construct a natural topology - the cone topology - on $\bar{\bar{H}}$ so that it is a closed disc with $H(\infty)$ as boundary. Before doing that we extend some of our previous definitions from H to $\bar{\bar{H}}$.

If $\gamma$ is a geodesic, let $\gamma(\infty)$ and $\gamma(-\infty)$ be the points at infinity to which $\gamma$ and $\bar{\gamma}$ belong respectively. It is clear from (6.3a) that any point in H and any point in $H(\infty)$ are joined by a unique (up to reparametrization) geodesic. If $p \in H$, $x \in \bar{\bar{H}}$ and $p \neq x$, let $\gamma_{px}$ be the geodesic with $\gamma_{px}(0) = p$ and $\gamma_{px}(t) = x$ for some $t \in (0,\infty]$. Let $V(p,x) = \dot{\gamma}_{px}(0)$. If $p \in H$, $x,y \in \bar{\bar{H}}$ and $x \neq p \neq y$, define $\measuredangle_p(x,y) = \measuredangle_p(V(p,x), V(p,y))$. These definitions are consistent with (4.9). Finally, extend the distance function d to $(\bar{\bar{H}}\times\bar{\bar{H}})\setminus(H(\infty)\times H(\infty))$ by defining $d(x,y) = \infty$ if x or y is in $H(\infty)$.

We can restate (6.3) in a form which will be convenient in Part B.

6.6 <u>LEMMA</u>

Suppose we have sequences $\{q_n\} \subseteq H$, $\{v_n\} \subseteq SH$ and $\{t_n\} \subseteq (0,\infty]$ with $q_n \to q$, $v_n \to v$ and $t_n \to \infty$ as $n \to \infty$. Then as $n \to \infty$ ,

$$V(q_n,\gamma_{v_n}(t_n)) \to V(q,\gamma_v(\infty)).$$

This is Theorem 5.1 of [39], except that Pesin has a lower bound on the curvature.

Proof

It follows from (6.3c) that

$$\measuredangle_{q_n}(\gamma_{v_n}(t_n),\gamma_{v_n}(\infty)) \to 0;$$

and from (6.2d) that

$$V(q_n,\gamma_{v_n}(\infty)) \to V(q,\gamma_v(\infty)). \qquad \square$$

Remark

There is a minor problem here because $V(q_n,\gamma_{v_n}(t_n))$ is not defined if $\gamma_{v_n}(t_n) = q_n$. There is no real difficulty, however, because this can happen for only finitely many n, since it is clear that $\gamma_{v_n}(t_n)$ eventually leaves any compact subset of H. Similar problems will be largely ignored in the rest of this section.

## B. THE CONE TOPOLOGY

Now we construct a natural topology which makes $\bar{H}$ a closed disc with $H(\infty)$ as its boundary. This was first done in the case we want to consider - when H has no focal points - by Goto [26]. We will not, however, use her construction. Instead we will follow the account in [14] of the construction originally used by Eberlein and O'Neill [19] in the case when H has non-positive curvature. With the help of (6.6) this goes through almost unchanged.

6.7 DEFINITION

If $p \in H$, $x \in H(\infty)$, $R > 0$ and $0 < \theta$ , the *truncated cone* with *vertex* p, *centre* x, *radius* R *and angle* $\theta$ is

$$C(p,x,R,\theta) = \{y \in \bar{\bar{H}}:d(p,y) > R \text{ and}$$

$$\sphericalangle_p(x,y) < \theta\} .$$

6.8 PROPOSITION [14, Proposition 1.12]

(i) The open sets of H and the truncated cones form a basis for a topology on $\bar{\bar{H}}$.

(ii) For any $p \in H$, the truncated cones with vertex p and the open sets of H are a basis for this topology.

Proof

The intersection of two open subsets of H and the intersection of a truncated cone with an open subset of H are both open in H. If S and T are two truncated cones, $H \cap S \cap T$ is open in H. Thus both (i) and (ii) will follow if we show that for any $p \in H$ and any $x \in S \cap T \cap H(\infty)$ there is a truncated cone with vertex p containing x and contained in $S \cap T$. By (6.9) below, there are truncated cones $S' \subseteq S$ and $T' \subseteq T$ with vertex p and centre x. Clearly $S' \cap T'$ is a truncated cone and $x \in S' \cap T' \subseteq S \cap T$. □

6.9 LEMMA

Suppose $p \in H$, $x \in H(\infty)$ and $T$ is a truncated cone containing $x$. Then there is a truncated cone with vertex $p$ and centre $x$ contained in $T$.

Proof

Let $p_o$ be the centre and $R_o$ the radius of $T$. We can choose $\theta_0 > 0$ so that the cone

$$C_o = C(p_o, x, R_o, \theta_o) \subseteq T.$$

It is clear from (6.6) (with $q_n \equiv p_o$, $\{v_n\} \subseteq S_pH$ and $v = V(p,x)$) that if $\theta > 0$ is small enough and $R$ is big enough,

$$\measuredangle_{p_o}(\gamma_w(t), x) < \theta_o$$

for all $t \geq R$ and all $w \in S_pH$ with $\measuredangle_p(w, V(p,x)) < \theta$.
If we also choose $R \geq d(p,p_o) + R_o$, we will have

$$C(p,x,R,\theta) \subseteq C_o \subseteq T. \quad \square$$

The topology constructed in (6.8) is called the *cone topology*. It is clearly Hausdorff. We now give a second description of the cone topology which will show that it makes $\bar{H}$ homeomorphic to a closed disc with $H(\infty)$ as boundary. For any $p \in H$ there is a natural way to add a boundary sphere to $T_pH$ and topologize the resulting space. Think of $T_pH$ as $S_pH \times [0,\infty)$ under the equivalence relation $\sim$ that identifies all the pairs $(u,0)$ such that $u \in S_pH$. Let $\bar{T}_pH$ be the quotient of $S_pH \times [0,\infty]$ under $\sim$ and give it the quotient topology. Let $[u,t]$ denote the $\sim$-equivalence class of $(u,t) \in S_pH \times [0,\infty]$. We can extend the exponential map to $\bar{T}_pH$ by defining

$$\mathrm{Exp}_p[u,t] = \gamma_u(t).$$

6.10 <u>PROPOSITION</u> [14, Proposition 1.13]

For any $p \in H$, $\mathrm{Exp}_p : \bar{T}_pH \to \bar{H}$ is a homeomorphism.

<u>Proof</u>

$\mathrm{Exp}|T_pH = \exp_p : T_pH \to H$, which is a diffeomorphism by (4.8). It is clear from this and (6.3a) that $\mathrm{Exp}_p$ is a bijection. Since $\bar{T}_pH$ is compact and $\bar{H}$ is Hausdorff we now have only to prove that $\mathrm{Exp}_p$ is continuous. By (6.8ii) it suffices to prove that the inverse images of open subsets of H and truncated cones with vertex p are open in $\bar{T}_pH$. But that is obvious. □

Since $\mathrm{Exp}_p^{-1}(x) = [V(p,x),\ d(p,x)]$ for any $x \in \bar{H}\setminus\{p\}$, we obtain immediately

6.11 <u>COROLLARY</u>

If $x_n \to x$ in $\bar{H}$ then $V(p,x_n) \to V(p,x)$ and $d(p,x_n) \to d(p,x)$

for every $p \in H \setminus \{x\}$. Conversely, if $p \in H \setminus \{x\}$ and $V(p,x_n) \to V(p,x)$ and $d(p,x_n) \to d(p,x)$, then $x_n \to x$ in $\bar{H}$.

6.12 <u>COROLLARY</u>

Suppose $\{p_n\}$ and $\{q_n\}$ are sequences in $H$ with $d(p_n,q_n)$ bounded. If $p_n \to x \in H(\infty)$, then $q_n \to x$.

<u>Proof</u>

Fix $p \in H$. By (6.11), $d(p,p_n) \to \infty$ and $V(p,p_n) \to V(p,x)$. Since $d(p_n,q_n)$ is bounded, $d(p,q_n) \to \infty$ , and $V(p,q_n) \to V(p,x)$ since $\measuredangle_p(p_n,q_n) \to 0$ by (5.14). It follows from (6.11) that $q_n \to x$. □

Now we show as in [14, 19] that the cone topology makes continuous various functions that ought to be continuous.

6.13 <u>PROPOSITION</u>

If we give $\bar{H}$ the cone topology, the following are continuous:

(i) $V: A = \{(p,x) \in H \times \bar{H} : p \neq x\} \to SH$;

(ii) $\measuredangle: \{(x,p,y) \in \bar{H} \times H \times \bar{H} : x \neq p \neq y\} \to [0,\pi]$;

(iii) $d: (\bar{H}\times\bar{H}) \setminus (H(\infty)\times H(\infty)) \to [0,\infty]$;

(iv) the map $SH \times [-\infty,\infty] \to \bar{H}$, $(v,t) \to \gamma_v(t)$.

<u>Proof</u>

(i) We want to show that if $(p_n,x_n) \to (p,x)$ in $A$, then $V(p_n,x_n) \to V(p,x)$. This is obvious if $x \in H$, so assume $x \in H(\infty)$. Then if $t_n = d(p,x_n)$, $t_n \to \infty = d(p,x)$, and by (6.11) if $v_n = V(p,x_n)$ and $v = V(p,x)$, then $v_n \to v$. Hence by (6.6)

$$V(p_n,x_n) = V(p_n,\gamma_{v_n}(t_n)) \to V(p,\gamma_v(\infty)) = V(p,x).$$

(ii) This follows from (i) since

$$\cos \measuredangle_p(x,y) = \langle V(p,x), V(p,y)\rangle.$$

(iii) Obvious.

(iv) We want to show that if $(v_n,t_n) \to (v,t)$ in $SH \times [-\infty,\infty]$, then $x_n \to x$ in $\bar{H}$, where $x_n = \gamma_{v_n}(t_n)$ and $x = \gamma_v(t)$. This is obvious if $t = 0$, so assume $t \neq 0$. Then $V(p,x)$ is defined and so is $V(p,x_n)$ for all except finitely many $n$. We see from (i) and (iii) that $V(p,x_n) \to V(p,x)$ and $d(p,x_n) \to d(p,x)$. It follows from (6.11) that $x_n \to x$. □

C. BUSEMANN FUNCTIONS AND HOROSPHERES

The geodesics which pass through a point $p \in H$ have a family of orthogonal hypersurfaces - the geodesic spheres with centre $p$. We shall see that the geodesics which end at a given point in $H(\infty)$ also have a family of orthogonal hypersurfaces - called horospheres. First we shall construct the so-called Busemann functions which measure the relative distance of points in $H$ from points in $H(\infty)$. The horospheres will be the level hypersurfaces of these functions. This subsection is based on [21]. For an alternative development of horospheres see §6 of [39].

Suppose $v \in SH$. For each $r \in \mathbb{R}$, define $b_{v,r}: H \to \mathbb{R}$, $b_{v,r}(p) = d(p,\gamma_v(r))-r$. We see from the triangle inequality that if $0 < r < s$, then $d(p,\gamma_v(r)) + s-r \geq d(p,\gamma_v(s))$ and $d(\gamma_v(0),p) + d(p,\gamma_v(s)) \geq s$. Hence if $0 < r < s$,

$$b_{v,r}(p) \geq b_{v,s}(p) \geq -d(p,\gamma_v(0)).$$

We see that as $r \to \infty$, $b_{v,r}$ converges pointwise to a function $b_v: H \to \mathbb{R}$, which is called the *Busemann function for* $v$. Note that $b_v(\gamma_v(t)) = -t$.

6.14 <u>PROPOSITION</u>

Every Busemann function $b_v$ is convex and $C^1$, and $\mathrm{grad}_p\, b_v = -V(p,\gamma_v(\infty))$.

<u>Proof</u>

By (4.14ii) each of the functions $b_{v,r}$ is convex, so $b_v$ is convex.

It is clear from its definition that $b_{v,r}$ has the same derivative as $d(\cdot,\gamma_v(r))$. We now see from (4.10i) that $\mathrm{grad}_p\, b_{v,r} = -V(p,\gamma_v(r))$ unless $p = \gamma_v(r)$. By (6.3a), $V(p,\gamma_v(r)) \to V(p,\gamma_v(\infty))$ as $r \to \infty$ uniformly for all $p$ in any given compact subset of $H$. Since $b_{v,r} \to b_v$ pointwise, it follows that $\mathrm{grad}_p\, b_v = -V(p,\gamma_v(\infty))$; this is a continuous function of $p$ by (6.3d). □

Eschenburg [21, Proposition 1] has proved that $b_v$ is $C^1$, assuming only that H has no conjugate points.

6.15 COROLLARY

If $v,w \in SH$, then $\gamma_v$ is asymptotic to $\gamma_w$ if and only if $b_v - b_w$ is constant.

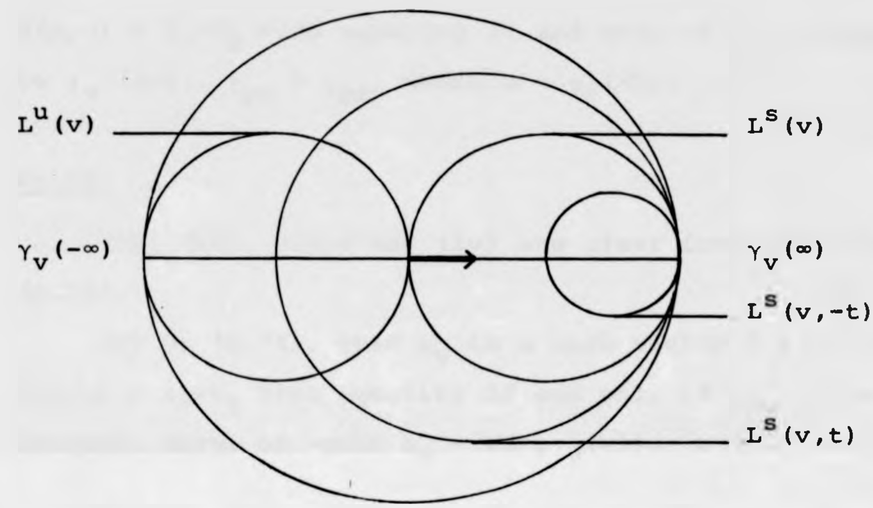
Proof

$\mathrm{grad}_p(b_v - b_w) = V(p,\gamma_w(\infty)) - V(p,\gamma_v(\infty)) = 0$ if and only if $\gamma_v(\infty) = \gamma_w(\infty)$. □

6.16 DEFINITION

A *horosphere* or *limiting sphere* is a level set of a Busemann function. If $v \in SH$, let $L^s(v,t) = \{p \in H : b_v(p) = t\}$. Let $L^u(v,t) = L^s(-v,-t)$, $L^s(v) = L^s(v,0)$ and $L^u(v) = L^u(v,0)$. If H has dimension 2, horospheres are often called *horocycles*.

The classical examples of horospheres are the horocycles in the Poincare disc. They are the (Euclidian) circles tangent to the boundary circle. Four of them are shown in the figure on the next page; t is a positive number.




Since grad $b_v$ never vanishes, we see that horospheres are embedded submanifolds.

6.17 PROPOSITION

(i) The horospheres $L^s(v,t)$ (resp. $L^u(v,t)$) are orthogonal to the geodesics positively (resp. negatively) asymptotic to $\gamma_v$. Each such geodesic meets each of these horospheres exactly once.

(ii) $L^s(v)$ and $L^u(v)$ are tangent at $\pi v$ and are both orthogonal to v.

(iii) If $\gamma$ and $\delta$ are asymptotic geodesics, then $\delta(0) \in L^s(\dot\gamma(0))$ if and only if $\gamma(0) \in L^s(\dot\delta(0))$.

(iv) $L^s(v,t) = L^s(\dot\gamma_v(s),\ t + s)$.

(v) If $t_1 > t_2$, $p \in L^s(v,t_1)$ and $q \in L^s(v,t_2)$, then

$d(p,q) \geq t_1 - t_2$ with equality if and only if $\gamma_{pq}$ is asymptotic to $\gamma_v$ (i.e., $\gamma_{pq} = \gamma_{px}$, where $x = \gamma_v(\infty)$).

Proof

(i), (ii), (iii) and (iv) are clear from (6.14) and (6.15).

(v) By (6.14), grad $b_v$ is a unit vector field. Hence $d(p,q) \geq t_1 - t_2$ with equality if and only if $\gamma_{pq}$ is an integral curve of $-\text{grad } b_v = V(\cdot, \gamma_v(\infty))$. □

We already know from (6.14) that horospheres are $C^1$ submanifolds of H. Now we show that they are $C^2$.

6.18 PROPOSITION [21, Theorem 1]

Every Busemann function $b_v$ is $C^2$.

This has also been proved in the case when H has non-positive curvature by Heintze and Im Hof [31] and Eberlein (unpublished, cited in [31]).

Proof

Consider a fixed $v \in SH$. As in (6.14), the idea is that the derivatives of $b_{v,r}$ converge to those of $b_v$. Clearly $b_{v,r}$ is $C^\infty$ on $H \setminus \{\gamma_v(r)\}$ and we saw in (6.14) that $b_v$ is $C^1$. Write $W(r,p) = \text{grad}_p\, b_{v,r}$ $(p \neq \gamma_v(r))$ and $W(p) = \text{grad}_p\, b_v$. We saw in (6.14) that $W(r,\cdot) \to W$ pointwise on H. Thus it will follow that $b_v$ is $C^2$ if we show that

$\nabla^2 b_{v,r}$ converges uniformly on any compact subset K of H.

It is clear from its definition that $b_{v,r}$ has the same first and second derivatives as $d(\cdot,\gamma_v(r))$. It follows using (4.10) that if $p \neq \gamma_v(r)$, then $W(r,p) = -V(p,\gamma_v(r))$. Also if $w \in T_pH$ and $w_1, w_2 \in W(r,p)^\perp$

$$\nabla^2 b_{v,r}(p)(w,W(r,p)) = 0$$

and

$$\nabla^2 b_{v,r}(p)(w_1,w_2) = \langle II(r,p)w_1,w_2\rangle,$$

where $II(r,p)$ is the second fundamental tensor of the sphere $S_{r,p} = S(\gamma_v(r), d(\gamma_v(r),p))$ relative to $W(r,p)$.

We see that $\nabla^2 b_{v,r}(p)$ will converge uniformly for $p \in K$ if we can show that $W(r,p)$ and $II(r,p)$ converge uniformly as $r \to \infty$ for $p \in K$. We saw in the proof of (6.14) that $W(r,\cdot) \to W$ uniformly on compact subsets of H. Since $\gamma_v(r)$ is the centre of $S_{r,p}$ and $W(r,p) = -V(p,\gamma_v(r))$ is the outward unit normal to $S_{r,p}$ at p, we see that $\gamma_{W(r,p)}(-t(r,p)) = \gamma_v(r)$, where $t(r,p)$ is the radius of $S_{r,p}$. It follows from (5.1) that

$$II(r,p) = D'_{-t(r,p)}(W(r,p),0). \qquad (*)$$

Since $-t(r,p) \to -\infty$ and $W(r,p) \to W(p)$ as $r \to \infty$ uniformly for $p \in K$, it follows from (*) and (5.9) that

$$II(r,p) \to D^{u'}(W(p),0)$$

uniformly for $p \in K$. $\square$

$\nabla^2 b_{v,r}$ converges uniformly on any compact subset K of H.

It is clear from its definition that $b_{v,r}$ has the same first and second derivatives as $d(\cdot,\gamma_v(r))$. It follows using (4.10) that if $p \neq \gamma_v(r)$, then $W(r,p) = -V(p,\gamma_v(r))$. Also if $w \in T_pH$ and $w_1, w_2 \in W(r,p)^\perp$

$$\nabla^2 b_{v,r}(p)(w,W(r,p)) = 0$$

and

$$\nabla^2 b_{v,r}(p)(w_1,w_2) = \langle II(r,p)w_1,w_2\rangle,$$

where $II(r,p)$ is the second fundamental tensor of the sphere $S_{r,p} = S(\gamma_v(r), d(\gamma_v(r),p))$ relative to $W(r,p)$.

We see that $\nabla^2 b_{v,r}(p)$ will converge uniformly for $p \in K$ if we can show that $W(r,p)$ and $II(r,p)$ converge uniformly as $r \to \infty$ for $p \in K$. We saw in the proof of (6.14) that $W(r,\cdot) \to W$ uniformly on compact subsets of H. Since $\gamma_v(r)$ is the centre of $S_{r,p}$ and $W(r,p) = -V(p,\gamma_v(r))$ is the outward unit normal to $S_{r,p}$ at p, we see that $\gamma_{W(r,p)}(-t(r,p)) = \gamma_v(r)$, where $t(r,p)$ is the radius of $S_{r,p}$. It follows from (5.1) that

$$II(r,p) = D'_{-t(r,p)}(W(r,p),0). \qquad (*)$$

Since $-t(r,p) \to -\infty$ and $W(r,p) \to W(p)$ as $r \to \infty$ uniformly for $p \in K$, it follows from (*) and (5.9) that

$$II(r,p) \to D^{u'}(W(p),0)$$

uniformly for $p \in K$. □

6.19 <u>COROLLARY</u>

Suppose $v \in SH$. If $p \in L^s(v)$, let $\hat{p}$ be the unit normal to $L^s(v)$ at $p$ on the same side as $v$. Define $\phi: L^s(v) \times \mathbb{R} \to H$,

$$\phi(p,t) = \gamma_{\hat{p}}(t).$$

Then $\phi$ is a normal family of hypersurfaces along $\gamma_v$ and $D^s(v,t)$ is related to $\phi$.

<u>Proof</u>

Since $L^s(v)$ is $C^2$ by (6.18), the argument at the end of §3B shows that $\phi$ is a normal family of hypersurfaces. That argument will also show that $D^s(v,t)$ is related to $\phi$ if we can show that $D^{s'}D^{s^{-1}}(v,0)$ is the second fundamental tensor, II, of $L^s(v)$ relative to $v$. Since $D^s(v,0) = I$, we need to show that $D^{s'}(v,0) = II$.

Since $\operatorname{grad}_{\pi v} b_v = -v$ (by 6.14), it is clear from the end of the previous proof that if $w_1, w_2 \in v^{\perp}$,

$$\begin{aligned}\nabla^2 b_v(\pi v)(w_1,w_2) &= \langle D^{u'}(-v,0)w_1, w_2\rangle \\ &= -\langle D^{s'}(v,0)w_1, w_2\rangle\end{aligned}$$

by (5.5ii). On the other hand, (6.14) says that if $p \in L^s(v)$, then $\operatorname{grad}_p b_v = -\hat{p}$ It is clear from this that

$$\nabla^2 b_v(\pi v)(w_1,w_2) = -\langle IIw_1, w_2\rangle. \quad \square$$

Now we show that the horospheres depend continuously on v.

6.20 PROPOSITION

The maps $(v,p) \to b_v(p)$, $(v,p) \to \operatorname{grad}_p b_v$ and $(v,p) \to \nabla^2 b_v(p)$ are continuous on $SH \times H$.

Proof

It follows from (6.14) and (6.3d) that $(v,p) \to -V(p,\gamma_v(\infty)) = \operatorname{grad}_p b_v$ is continuous on $SH \times H$.

We see from the proof of (6.18) that for all $w \in T_pH$ and $w_1,w_2 \in (\operatorname{grad} b_v(p))^{\perp}$, we have

$$\nabla^2 b_v(p)(w,\operatorname{grad}_p b_v) = 0$$

and

$$\nabla^2 b_v(p)(w_1,w_2) = \langle D^{u'}(\operatorname{grad}_p b_v,0)w_1,w_2\rangle.$$

Since $(v,p) \to \operatorname{grad}_p b_v$ is continuous and $u \to D^{u'}(u,0)$ is continuous on $SH$ (by 5.9), we see that $(v,p) \to \nabla^2 b_v(p)$ is continuous.

To see that $(v,p) \to b_v(p)$ is continuous, note that

$$b_v(p) = \int_0^1 \langle \operatorname{grad}_{\sigma(s)} b_v, \dot{\sigma}(s)\rangle ds,$$

where

$$\begin{aligned} \sigma(s) &= p \text{ for all } s \text{ if } p = \pi(v), \\ &= \gamma_{\pi(v)p}(s.d(p,\pi(v))) \text{ otherwise.} \end{aligned}$$

Since the integrand depends continuously on $(v,p)$, so does $b_v(p)$. □

## D. BIASYMPTOTIC GEODESICS

Suppose $v \in SH$. A point p in SH lies on a geodesic biasymptotic to $\gamma_v$ if and only if $V(p,\gamma_v(-\infty)) = -V(p,\gamma_v(\infty))$. We see from (6.14) that this occurs if and only if p is a critical point of $b_v + b_{-v}$.

6.21 LEMMA

(i) $b_v + b_{-v}$ is convex.

(ii) $b_v + b_{-v} \geq 0$.

(iii) p is a critical point of $b_v$ if and only if $b_v(p) + b_{-v}(p) = 0$.

(iv) $b_v(p) + b_{-v}(p) = 0$ if and only if p lies on a geodesic biasymptotic to $\gamma_v$.

Proof

(i) $b_v$ and $b_{-v}$ are convex by (6.14).

(ii) For any p in H and any r, we have

$$b_{v,r}(p) + b_{-v,r}(p) = d(p,\gamma_v(r)) + d(p,\gamma_{-v}(r)) - 2r$$
$$\geq 0$$

by the triangle inequality. Now let $r \to \infty$ .

(iii) This follows from (i) and (ii), since $b_v + b_{-v}$ attains the value 0 at $\pi v$.

(iv) This is clear from (iii) and the remarks before the lemma. □

6.22 DEFINITION

If $v \in SH$, then $B(v) = L^s(v) \cap L^u(v)$.

6.23 PROPOSITION

(i) $B(v)$ is convex and closed.

(ii) $L^s(v)$ and $L^u(v)$ are tangent at points of $B(v)$ and their common normals are biasymptotic to $\gamma_v$. These are the only geodesics biasymptotic to $\gamma_v$.

(iii) If $\gamma$ and $\delta$ are biasymptotic geodesics in H and $\delta(0) \in B(\dot\gamma(0))$, then $B(\dot\gamma(t)) = B(\dot\delta(t))$ for any t.

(iv) If $v_n \to v$ in SH, $p_n \to p$ in H and $p_n \in B(v_n)$ for each n, then $p \in B(v)$.

Proof

(i) By its definition, $B(v) = \{p \in H : b_v(p) = 0 = b_{-v}(p)\}$. It is clear from this and (6.21ii) that $B(v) = \{p \in H : b_v(p) \le 0$ and $b_{-v}(p) \le 0\}$, which is closed and convex since $b_v$ and $b_{-v}$ are continuous and convex by (6.14).

(ii) If $p \in B(v)$, we have $b_v(p) + b_{-v}(p) = 0$, so p lies on a geodesic biasymptotic to $\gamma_v$ by (6.21iv). This geodesic is orthogonal to both $L^s(v)$ and $L^u(v)$ by (6.17i).

Now suppose $\delta$ is a geodesic biasymptotic to $\gamma_v$. By (6.17i), $\delta$ meets $L^s(v)$ in a point, p say. By (6.21iv), $b_v(p) + b_{-v}(p) = 0$. Since $b_v(p) = 0$, we have $b_{-v}(p) = 0$ also. Thus $p \in L^s(v) \cap L^u(v) = B(v)$. It is clear from (6.17i) that $\delta$ is orthogonal to $L^s(v)$ and $L^u(v)$.

(iii) This is obvious from (ii) and (6.17).

(iv) We have $b_{v_n}(p_n) = 0 = b_{-v_n}(p_n)$ for every n. Since $(w,q) \to b_w(q)$ is continuous (6.20), we obtain $b_v(p) = 0 = b_{-v}(p)$. Thus $p \in B(v)$. □

The next result is known for obvious reasons, as "the Flat Strip Theorem". Suppose $\gamma$ and $\delta$ are biasymptotic geodesics, and choose the parametrization of $\delta$ so that $\delta(0) \in B(\dot\gamma(0))$.

6.24 <u>PROPOSITION</u>

Let $\gamma$ and $\delta$ be as above and let $c = d(\gamma(0),\delta(0))$. There is a totally geodesic isometric embedding

$F:[0,c] \times \mathbb{R} \to H$ such that $F(0,t) \equiv \gamma(t)$ and $F(c,t) \equiv \delta(t)$. Of course, $[0,c] \times \mathbb{R}$ has the flat Euclidean metric.

This was proved by Eberlein and O'Neill [19, Proposition 5.1] when H has non-positive curvature, and when H has no focal points by O'Sullivan [37, Theorem 1] and Eschenburg [21, Theorem 2iii]. I give Eschenburg's proof.

<u>Proof</u>

We can assume that $c > 0$, since the case $c = 0$ is trivial. Let $\sigma_t = \gamma_{\gamma(t)\delta(t)}$. We see from (iii) and (i) of (6.23) that $B(\dot\gamma(t)) = B(\dot\delta(t))$ and this set contains the geodesic segment $\sigma_t\,[0,c]$. It follows from (6.23ii) that for each $s \in [0,c]$ there is a unique geodesic $\beta_s$ with

$\beta_s(0) = \sigma_0(s)$ that is biasymptotic to $\gamma$ and $\delta$.

We now show that $\beta_s(t) \equiv \sigma_t(s)$. It follows from (6.4iii) that for any $r, s \in [0,c]$

$$\begin{aligned} d(\beta_r(t), \beta_s(t)) &\equiv d(\beta_r(0),\beta_s(0)) \\ &= d(\sigma_0(r),\sigma_0(s)) \\ &= |r-s|, \end{aligned}$$

since $\sigma_0$ is a geodesic. It follows that for any fixed t, the map $[0,c] \to H$, $s \to \beta_s(t)$, is a geodesic segment. Since this geodesic joins $\gamma(t)$ to $\delta(t)$, it is clear that $\beta_s(t) = \sigma_t(s)$.

Now define $F:[0,c] \times \mathbb{R} \to H$ by

$$F(s,t) = \beta_s(t) = \sigma_t(s).$$

We show that F has the desired properties. It is clear that $(s,t) \to \sigma_t(s)$ is $C^\infty$ and $(s,t) \to \beta_s(t)$ is injective. Since $\beta_s$ and $\sigma_t$ are geodesics we see that

$$\left\|\frac{\partial F}{\partial s}\right\| \equiv 1 \equiv \left\|\frac{\partial F}{\partial t}\right\| \quad , \tag{1}$$

and

$$\frac{D}{ds}\frac{\partial F}{\partial s} \equiv 0 \equiv \frac{D}{dt}\frac{\partial F}{\partial t}. \tag{2}$$

It is clear from (6.19) that for any $s_o \in [0,c]$, the Jacobi field $\frac{\partial F}{\partial s}(s_o,t)$ along $\beta_{s_o}$ is both stable and unstable and hence orthogonal and (by 5.12iii) covariantly constant. Thus

$$\langle \frac{\partial F}{\partial s}, \frac{\partial F}{\partial t} \rangle \equiv 0, \tag{3}$$

and

$$\frac{D}{dt} \frac{\partial F}{\partial s} \equiv 0. \tag{4}$$

It follows from (1) and (3) that F is an isometry and from (2) and (4) that F is totally geodesic. □

6.25 DEFINITION

A geodesic $\gamma$ in H bounds a *flat strip of width* c if there is a totally geodesic isometric embedding

$$F: [0,c] \times \mathbb{R} \to H \text{ with } F(0,\cdot) = \gamma.$$

If there is a map $F: [0,\infty) \times \mathbb{R} \to H$ with similar properties, then $\gamma$ bounds a *flat half plane*.

6.26 REMARKS

Clearly $\gamma$ bounds a flat half plane if and only if $B(\dot{\gamma}(0))$ is non-compact. Also if $\gamma$ and $\delta$ are biasymptotic, then $\gamma$ bounds a flat half plane if and only if $\delta$ does. It is clear that the map $B(\dot{\gamma}(0)) \times \mathbb{R} \to H$,

$$(p,t) \to \gamma_{p\gamma(\infty)}(t)$$

is an isometry of $B(\dot{\gamma}(0)) \times \mathbb{R}$ onto the set of points in H that lie on geodesics biasymptotic to $\gamma$.

## E. ANGLES, CONES AND FLAT HALF PLANES

Suppose $\gamma$ is a geodesic in $H$ and $x$ is a point in $\bar{H}$. We study the angle $\theta(t) = \measuredangle_{\gamma(t)}(x,\gamma(\infty))$.

6.27 PROPOSITION

$\theta(t)$ is always non-decreasing, and is strictly increasing if $x \in H$ and $x$ does not lie on $\gamma$.

Proof

If $x = \gamma(s)$ for some $s \in [-\infty,\infty]$, then $\theta(t) = 0$ for $t < s$ and $\theta(t) = \pi$ for $t > s$. It was proved in (4.16) that $\theta$ is strictly increasing if $x$ lies in $H$ and not on $\gamma$.

Now suppose $x \in H(\infty) \setminus \{\gamma(-\infty),\gamma(\infty)\}$. Choose a sequence $\{x_n\} \subseteq H$ such that no $x_n$ lies on $\gamma$ and $x_n \to x$. Let $\theta_n(t) = \measuredangle_{\gamma(t)}(x_n,\gamma(\infty))$. Then $\theta_n$ is strictly increasing by the above and, for each $t$, $\theta_n(t) \to \theta(t)$ by (6.13ii). Hence $\theta$ is non-decreasing. $\square$

There is an immediate application to cones.

6.28 COROLLARY

If $t_1 > t_2$ and $0 < \theta_1 < \theta_2$,

$$C(\gamma(t_1),\gamma(\infty),0,\theta_1) \subseteq C(\gamma(t_2),\gamma(\infty),0,\theta_2).$$

See (6.7) for the notation.

We now consider the extreme case in which the angle $\theta(t)$ is constant.

6.29 PROPOSITION

Suppose $\gamma$ is a geodesic in H and $x \in \bar{H} \setminus \{\gamma(-\infty), \gamma(\infty)\}$. Suppose $\theta(t) = \measuredangle_{\gamma(t)}(x, \gamma(\infty))$ is independent of t. Then $\gamma$ bounds a flat, totally geodesically embedded half plane: namely the image of the map $F: [0,\infty) \times \mathbb{R} \to H$,

$$F(s,t) = \gamma_{\gamma(t)x}(s).$$

Proof

It is clear from (6.27) that $x \in H(\infty)$.

The main idea is to show that the curves $F(s_o, \cdot)$ are geodesics.

LEMMA

Suppose $p = F(s_o, t_o)$. Then

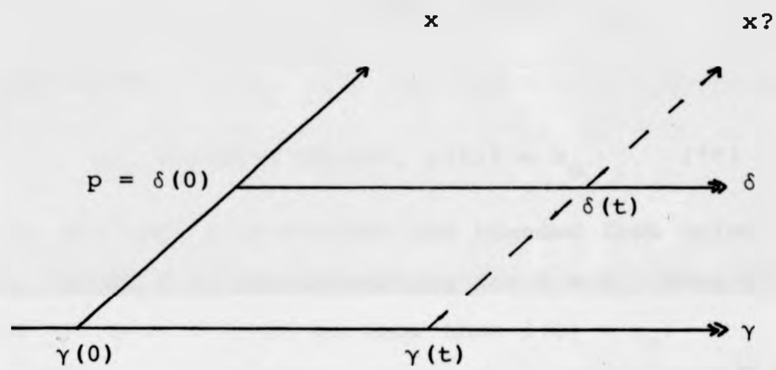
$$F(s_o, t+t_o) = \gamma_{p\gamma(\infty)}(t)$$

for all $t > 0$.

Proof

Assume for simplicity that $t_o = 0$. Write $\delta = \gamma_{p\gamma(\infty)}$. Then we want to show that for any $t > 0$,

$$\delta(t) = \gamma_{\gamma(t)x}(s_o).$$




Let $b = b_v$ be the Busemann function for a vector v with $\gamma_v(\infty) = x$.

We shall show that for $t > 0$

$$b(\gamma(t)) - b(\delta(t)) = d(\gamma(t),\delta(t)) = s_o. \quad (*)$$

Write $f(t) = b(\gamma(t)) - b(\delta(t))$. Note firstly that f is a *concave* function. This is because $(b\circ\delta)'' \geq 0$ since b is convex (6.14); and $(b\circ\gamma)'' \equiv 0$, since (by 6.14)

$$(b\circ\gamma)'(t) = \langle -V(\gamma(t),x),\ \dot{\gamma}(t)\rangle = -\cos\theta(t)$$

which is constant.

Since grad b is a unit vector field (see 6.14) we have

$$|f(t)| \leq d(\gamma(t),\ \delta(t)).$$

Since $\gamma$ and $\delta$ are asymptotic, it follows from (6.4i) that for $t > 0$

$$d(\gamma(t), \delta(t)) < d(\gamma(0), \delta(0)) = s_o.$$

Thus if $t > 0$,

$$-s_o < f(t) < d(\gamma(t), \delta(t)) < s_o. \qquad (**)$$

We see that f is concave and bounded from below for $t > 0$. Hence f is non-decreasing for $t > 0$. Thus we will obtain (*) from (**) if we show that $f(0) = s_o$.

Since $\gamma_{\gamma(0)\delta(0)}(\infty) = x$, it is clear from (6.14) that $\gamma_{\gamma(0)\delta(0)}$ is an integral curve of -grad b. Thus this curve is orthogonal to the horospheres defined by b. We see from (6.17v) that

$$b(\gamma(0)) - b(\delta(0)) = d(\gamma(0),\delta(0)).$$

Thus

$$f(0) = s_o,$$

and we have finally proved (*).

It follows from (*) and (6.17v) that for $t > 0$,

$$\gamma_{\gamma(t)\delta(t)} = \gamma_{\gamma(t)x}.$$

Hence for $t > 0$,

$$\begin{aligned}\delta(t) &= \gamma_{\gamma(t)\delta(t)}(d(\gamma(t),\delta(t))) \\ &= \gamma_{\gamma(t)x}(s_o). \quad \square\end{aligned}$$

By applying the lemma with values of $t_o$ tending to $-\infty$, we see that for any fixed $s_o > 0$ the curve $F(s_o,\cdot)$ is a geodesic. Clearly $d(\gamma(t), F(s_o,t)) \equiv s_o$. It follows from the Flat Strip Theorem (6.24) that $\gamma$ and $F(s_o,\cdot)$ bound a flat strip. This flat strip must contain all the geodesic segments joining $\gamma(t)$ to $F(s_o,t)$, so we see that it is the image of $F|[0,s_o] \times \mathbb{R}$. Since this is true for any $s_o > 0$, we see that im (F) is a flat half plane bounded by $\gamma$. $\square$

<u>Remarks</u>

(i) A sharper result holds if H has non-positive curvature [5, Proposition 2.1].

(ii) This answers a question raised by Ballmann at the end of his thesis [4, p. 54].

By applying the lemma with values of $t_o$ tending to $-\infty$, we see that for any fixed $s_o > 0$ the curve $F(s_o,\cdot)$ is a geodesic. Clearly $d(\gamma(t), F(s_o,t)) \equiv s_o$. It follows from the Flat Strip Theorem (6.24) that $\gamma$ and $F(s_o,\cdot)$ bound a flat strip. This flat strip must contain all the geodesic segments joining $\gamma(t)$ to $F(s_o,t)$, so we see that it is the image of $F|[0,s_o] \times \mathbb{R}$. Since this is true for any $s_o > 0$, we see that im (F) is a flat half plane bounded by $\gamma$. $\square$

<u>Remarks</u>

(i) A sharper result holds if H has non-positive curvature [5, Proposition 2.1].

(ii) This answers a question raised by Ballmann at the end of his thesis [4, p. 54].

## §7. AXIAL ISOMETRIES

This section is a preparation for §8 where we will establish some dynamical properties of the geodesic flow on a Riemannian manifold with no focal points. They will be obtained by studying the action of the group of covering transformations on the universal cover and its boundary sphere. Here we present the necessary information, mostly results of Ballmann, about isometries.

In this section H will be a simply connected Riemannian manifold with no focal points.

### A. ISOMETRIES

Any isometry $\phi$ of H can be extended to a homeomorphism $\bar{\phi}$ of $\bar{H}$ as follows. If $x \in H(\infty)$ and $\gamma$ is a geodesic with $\gamma(\infty) = x$, then $\bar{\phi}(x) = (\phi\circ\gamma)(\infty)$. This is valid because if $\gamma$ and $\delta$ are asymptotic geodesics, so are $\phi\circ\gamma$ and $\phi\circ\delta$. It is clear from (6.10) that $\bar{\phi}:\bar{H} \to \bar{H}$ is a homeomorphism. Also it is clear that $\overline{\phi_1 \circ \phi_2} = \bar{\phi}_1 \circ \bar{\phi}_2$. Henceforth we shall drop the bar and use $\phi$ to denote its extension to $\bar{H}$. It is clear that for any Busemann function $b_v$ we have $b_v = b_{\phi_* v} \circ \phi$. Thus $\phi$ maps horospheres to horospheres.

#### DEFINITION

An isometry $\phi$ of H is *axial* if it translates a geodesic of H, i.e. if there is a geodesic $\alpha$ and a number $\tau > 0$ such that $\phi(\alpha(t)) = \alpha(t+\tau)$ for all t. We call $\alpha$ an *axis* and $\tau$

the *period* of $\phi$ along $\alpha$. Note that $\alpha(-\infty)$ and $\alpha(\infty)$ are fixed points of $\phi$.

Axial isometries can have more than one axis; for example a translation of the Euclidean plane has infinitely many parallel axes. We do, however, have:

7.1 <u>PROPOSITION</u>

If $\phi$ is an axial isometry of H, then $\phi$ has the same period along any axis and all its axes are biasymptotic.

<u>Proof</u>

If $\tau$ is the period of $\phi$ along some axis $\alpha$, then $\phi_*(\dot{\alpha}(t)) \equiv \dot{\alpha}(t+\tau)$, and hence $\phi(L(\dot{\alpha}(t))) = L(\dot{\alpha}(t+\tau))$. We see from (6.17v) that for any $p \in H$, $d(p,\phi p) \geq \tau$. It follows that $\tau = \inf\{d(p,\phi p) : p \in H\}$; this depends only on $\phi$, so all axes of $\phi$ must have the same period $\tau$.

Now suppose that $\beta$ is another axis of $\phi$. For any integer n, $d(\alpha(n\tau),\beta(n\tau)) = d(\phi^n\alpha(0),\phi^n\beta(0)) = d(\alpha(0),\beta(0))$. It follows easily that $\alpha$ and $\beta$ are biasymptotic. □

Suppose $\phi$ is an isometry of H with axis $\alpha$ and period $\tau$ along $\alpha$. We see from the above argument that $\phi$ tends to "push" H along $\alpha$. Indeed, *for any* $p \in H, \phi^n p \to \alpha(\infty)$ *and* $\phi^{-n}p \to \alpha(-\infty)$ *as* $n \to \infty$. This follows from (6.12), since $d(\phi^n p,\phi^n\alpha(0))$ is independent of n, and $\phi^n\alpha(0) = \alpha(n\tau) \to \alpha(\pm\infty)$ as $n \to \pm\infty$.

One would like this behaviour to be reflected in the action of $\phi$ on $H(\infty)$: namely, if $x \in H(\infty) \setminus \{\alpha(-\infty), \alpha(\infty)\}$, one wants $\phi^n x \to \alpha(\infty)$ and $\phi^{-n} x \to \alpha(-\infty)$ as $n \to \infty$ . This is a classical property of axial isometries of the Poincaré disc. On the other hand, a translation of Euclidean space fixes every point at infinity. Ballmann [4,5] discovered that an axial isometry has the desired behaviour on $H(\infty)$ if and only if it is hyperbolic according to the following.

7.2 DEFINITION

An axial isometry is *hyperbolic* if it has an axis that does not bound a flat half plane.

Ballmann also showed that an axial isometry is hyperbolic if and only if there are geodesics joining the ends of its axes to all other points in $H(\infty)$. This will be needed in §8.

The rest of this section is an exposition of these results of Ballmann. He assumed that H has non-positive curvature. The proofs given here require only that H have no focal points. Two main changes are needed. Firstly, where Ballmann uses Proposition 1.2 of [5], we use (6.29), which is due to the present author. Secondly, the original proof of the next lemma must be replaced with a new argument, due to Ballmann.

## B. GEODESICS WITH NO FLAT HALF PLANE

7.3 LEMMA [4, Lemma 2.1, 5, (*) on p. 133]

Suppose $\gamma$ is a geodesic in H and $\{p_n\}, \{q_n\} \subseteq H \setminus \{\gamma(0)\}$ are sequences such that $p_n \neq q_n$ for each n, $\measuredangle_{\gamma(0)}(p_n, \gamma(-\infty)) \to 0$ and $\measuredangle_{\gamma(0)}(q_n, \gamma(\infty)) \to 0$. Suppose

$$d(\gamma(0), \gamma_{p_n q_n}) > c$$

for any n. Then $\gamma$ bounds a flat strip of width $> c$.

Proof

I thank Werner Ballmann for communicating to me the main idea in the following argument.

Write $\delta_n = \gamma_{p_n q_n}$.

First we show that $p_n \to \gamma(-\infty)$. This will follow from (6.11) if we show that $d(\gamma(0), p_n) \to \infty$ as $n \to \infty$. In the triangle $p_n, \gamma(0), q_n$ the angle at $\gamma(0)$ approaches $\pi$ as $n \to \infty$. It now follows from (4.18) that as $n \to \infty$, $\measuredangle_{p_n}(\gamma(0), q_n) \to 0$. But $\measuredangle_{p_n}(\gamma(0), q_n)) > \measuredangle_{p_n}(B(\gamma(0), c))$, since otherwise $\delta_n$ would pass within distance c of $\gamma(0)$. Thus $\measuredangle_{p_n}(B(\gamma(0), c)) \to 0$, which is impossible unless $d(\gamma(0), p_n) \to 0$.

A similar argument shows that $q_n \to \gamma(\infty)$.

Now we move $p_n$ and $q_n$ so that we have $d(\gamma(0), \delta_n) = c$ for all n. We move them so that $d(\gamma(0), p_n)$ and $d(\gamma(0), q_n)$ stay constant, and $\theta_n = \measuredangle_{\gamma(0)}(p_n, \gamma(-\infty))$ and $\psi_n = \measuredangle_{\gamma(0)}(q_n, \gamma(\infty))$

decrease. It is clear from (6.10) that we will still have $p_n \to \gamma(-\infty)$ and $q_n \to \gamma(\infty)$. By ignoring finitely many terms, we can assume that initially $\theta_n$ and $\psi_n$ are both $< \pi/2$ for every n. Then $p_n$ and $q_n$ will not meet as we move them, and so $d(\gamma(0),\delta_n)$ will change continuously. Since we start with $d(\gamma(0),\delta_n) > c$ and would end with $d(\gamma(0),\delta_n) = 0$ when $\theta_n = 0 = \psi_n$, it is clear that we can obtain $d(\gamma(0),\delta_n) = c$ at some stage.

Now by (4.17) there is, for each n, a unique $t_n$ such that

$$d(\gamma(0),\delta_n(t_n)) = d(\gamma(0),\delta_n) = c.$$

By passing to a subsequence if necessary, we can assume that $\delta_n(t_n)$ converges, to a say. Since $p_n \to \gamma(-\infty)$ and $q_n \to \gamma(\infty)$ we see from (6.13i) that

$$V(a,\gamma(\infty)) = \lim_{n\to\infty} V(\delta_n(t_n),q_n) = \lim_{n\to\infty} \dot{\delta}_n(t_n)$$

and

$$V(a,\gamma(-\infty)) = \lim_{n\to\infty} V(\delta_n(t_n),p_n) = \lim_{n\to\infty} -\dot{\delta}_n(t_n).$$

Let $\delta$ be the geodesic with $\dot{\delta}(0) = \lim_{n\to\infty} \dot{\delta}_n(t_n)$. It is clear that $\delta$ is biasymptotic to $\gamma$. Since $\delta_n(t_n)$ is the point on $\delta_n$ closest to $\gamma(0)$, it is clear that

$$d(\gamma(0),\delta) = d(\gamma(0),a) = c.$$

It is now clear from the Flat Strip Theorem (6.24) that $\gamma$ and $\delta$ bound a flat strip of width c. □

7.4 <u>PROPOSITION</u> [5, Lemma 2.1]

Suppose $\gamma$ is a geodesic in H that does not bound a flat half plane. For $\varepsilon > 0$, let $X_\varepsilon$ and $Y_\varepsilon$ be the closures in the cone topology of $C(\gamma(0),\gamma(-\infty),0,\varepsilon)$ and $C(\gamma(0),\gamma(\infty),0,\varepsilon)$ respectively. Let c be greater than the width of the widest flat strip bounded by $\gamma$. Then for any small enough $\varepsilon > 0$ we have the following.

(i) If $x \in X_\varepsilon$ and $y \in Y_\varepsilon$, then there is a geodesic joining x and y.

(ii) Let $\delta$ be a geodesic which joins $x \in X_\varepsilon \setminus \{\gamma(0)\}$ to $y \in Y_\varepsilon \setminus \{\gamma(0)\}$. Then $\delta$ passes within distance c of $\gamma(0)$ and does not bound a flat strip.

(iii) Suppose $\{p_n\}, \{q_n\} \subseteq H$ and $p_n \to \gamma(-\infty)$, $q_n \to \gamma(\infty)$. Then $\measuredangle_{p_n}(Y_\varepsilon) \to 0$ and $\measuredangle_{q_n}(X_\varepsilon) \to 0$ .

We have added (iii) for use in (8.5).

<u>Proof</u>

It is clear from (7.3) that for any small enough $\varepsilon > 0$ we have the following: $d(\gamma(0),\gamma_{pq}) < c$ whenever $p,q \in H \setminus \{\gamma(0)\}$, $\measuredangle_{\gamma(0)}(p,\gamma(-\infty)) < 2\varepsilon$ and $\measuredangle_{\gamma(0)}(q,\gamma(\infty)) < 2\varepsilon$. We now show that (i), (ii) and (iii) hold when $\varepsilon$ has this property.

(i) Choose sequences $\{p_n\} \subseteq C(\gamma(0),\gamma(-\infty),0,\varepsilon)$ and $\{q_n\} \subseteq C(\gamma(0),\gamma(\infty),0,\varepsilon)$ such that $p_n \to x$ and $q_n \to y$. It is clear from the choice of $\varepsilon$ that each $\gamma_{p_n q_n}$ passes within distance c of $\gamma(0)$. The last paragraph of the proof of (7.3) can now be adapted to construct a geodesic joining x to y.

(ii) It is obvious from (6.13) that we can choose points $p,q \in H\setminus\{\gamma(0)\}$ which lie on $\delta$ and have

$$\measuredangle_{\gamma(0)}(p,\gamma(-\infty)) < 2\varepsilon, \ \measuredangle_{\gamma(0)}(q,\gamma(\infty)) < 2\varepsilon.$$

We see from the choice of $\varepsilon$ that $\delta$ must pass within distance c of $\gamma(0)$. Similarly any geodesic biasymptotic to $\delta$ also passes within distance c of $\gamma(0)$. Hence $\delta$ cannot bound a flat strip of width greater than 2c.

(iii) We show that $\measuredangle_{p_n}(Y_\varepsilon) \to 0$. A similar argument shows that $\measuredangle_{q_n}(X_\varepsilon) \to 0$.

We show firstly that if n is large enough, then $\gamma_{p_n y}$ passes within distance c of $\gamma(0)$ for all $y \in Y_\varepsilon$. It is obvious (from 6.13) that for any n and any $y \in Y_\varepsilon$ there is a point $q_{ny} \in H\setminus\{\gamma(0)\}$ which lies on $\gamma_{p_n y}$ and has $\measuredangle_{\gamma(0)}(q_{ny},\gamma(\infty)) < 2\varepsilon$. Since $p_n \to \gamma(-\infty)$, it follows from (6.11) that for all large enough n we have $\measuredangle_{\gamma(0)}(p_n,\gamma(-\infty)) < 2\varepsilon$. It follows that for such n, $d(\gamma(0),\gamma_{p_n y}) < c$ for all $y \in Y_\varepsilon$.

Thus for all large enough n, $\measuredangle_{p_n}(Y_\varepsilon) < \measuredangle_{p_n}(B(\gamma(0),c))$. Since $d(p_n,\gamma(0)) \to \infty$, it is clear from (5.14) that the latter angle approaches 0 as $n \to \infty$. □

## C. HYPERBOLIC AXIAL ISOMETRIES

**7.5 THEOREM** [5, Theorem 2.2]

Suppose $\alpha$ is an axis of an isometry $\phi$ of H. The following are equivalent.

(i) $\alpha$ does not bound a flat half plane.

(ii) No axis of   bounds a flat half plane.

(iii) For any open sets $U,V \subseteq H(\infty)$ with $\alpha(-\infty) \in U$ and $\alpha(\infty) \in V$, we have $\phi^n(H(\infty)\setminus U) \subseteq V$ and $\phi^{-n}(H(\infty)\setminus V) \subseteq U$ for all large enough n.

(iv) Any $y \in H(\infty)\setminus\{\alpha(-\infty)\}$ is joined to $\alpha(-\infty)$ by a geodesic which does not bound a flat half plane. Any $x \in H(\infty)\setminus\{\alpha(\infty)\}$ is joined to $\alpha(\infty)$ by a geodesic which does not bound a flat half plane.

Proof

(i) $\Rightarrow$ (ii). Obvious from (7.1) and (6.26).

(ii) $\Rightarrow$ (i). Trivial.

Let $\tau$ be the period of $\phi$ along $\alpha$. For $z \in H(\infty)$, let $\theta_n(z) = \measuredangle_{\alpha(0)}(\phi^n z, \alpha(\infty))$. We see that

$$\theta_n(z) = \measuredangle_{\phi^{-n}\alpha(0)}(z, \phi^{-n}\alpha(\infty)) = \measuredangle_{\alpha(-n\tau)}(z, \alpha(\infty)). \qquad (*)$$

(iii) $\Rightarrow$ (i). Suppose $\alpha$ does bound a flat half plane. Let $\beta$ be the geodesic in the half plane with $\beta(0) = \alpha(0)$ and $\dot\beta(0) \perp \dot\alpha(0)$. It is obvious from Euclidean geometry that

$\measuredangle_{\alpha(t)}(\beta(\infty),\alpha(\infty)) = \pi/2$ for all t. We see from (*) that $\theta_n(\beta(\infty)) = \pi/2$ for all n. Hence $\phi^n\beta(\infty)$ does not converge to $\alpha(\infty)$, contrary to (iii).

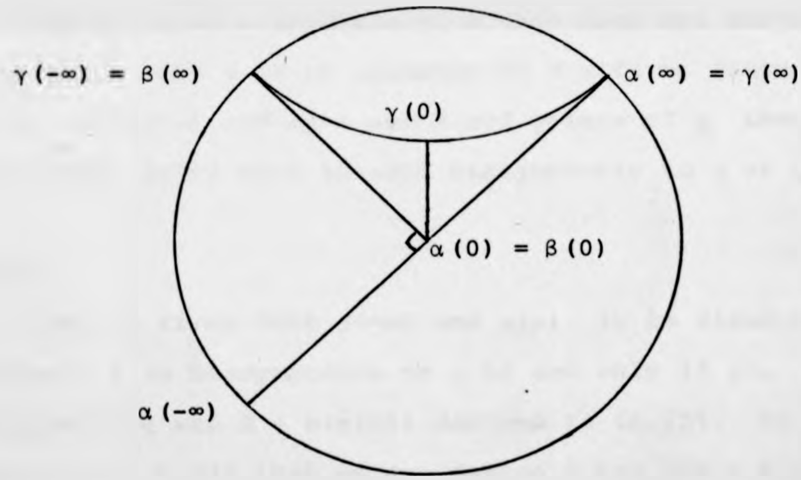
(i) $\Rightarrow$ (iii) We show that $\phi^n(H(\infty) \setminus U) \subseteq V$ for all large enough n; the proof of the other claim is similar. We see from (*) and (6.27) that the sequence $\{\theta_n(x)\}$ is non-increasing for every $x \in H(\infty)$. Let $\theta(x) = \lim\limits_{n\to\infty} \theta_n(x)$. We show that $\theta(x) = 0$ or $\pi$. Since $H(\infty)$ is compact, $\{\phi^r x : r > 1\}$ has a convergent subsequence, with limit y say. Since $\theta_n$ is non-increasing, $\theta_n(y) = \lim\limits_{r\to\infty} \theta_{r+n}(x) = \theta(x)$ for every integer n. We see from (*) that $\measuredangle_{\alpha(-n\tau)}(y,\alpha(\infty))$ is independent of n, so $\measuredangle_{\alpha(t)}(y,\alpha(\infty))$ is constant by (6.27). Since $\alpha$ does not bound a flat half plane, it follows from (6.29) that $y = \alpha(-\infty)$ or $\alpha(\infty)$. It follows easily that $\theta(x) = 0$ or $\pi$, and $\theta(x) = \pi$ only if $x = \alpha(-\infty)$.

Thus as $n \to \infty$, $\theta_n(x)$ decreases to 0 for all $x \in H(\infty)\setminus\{\alpha(-\infty)\}$. It is clear from (6.13ii) that each $\theta_n$ is continuous. We see from Dini's theorem that $\theta_n \to 0$ uniformly on compact subsets of $H(\infty)\setminus\{\alpha(-\infty)\}$, in particular on $H(\infty)\setminus U$. It is clear from this that $\phi^n(H(\infty)\setminus U)$ is eventually contained in any given neighbourhood of $\alpha(\infty)$.

(i) $\Rightarrow$ (iv). We prove the first claim. The proof of the second is similar. By (7.4) there is $\varepsilon > 0$ such that any $z \in H(\infty)$ with $\measuredangle_{\alpha(0)}(z,\alpha(\infty)) < \varepsilon$ is joined to $\alpha(-\infty)$ by a geodesic which does not bound a flat half plane. If

$y \in H(\infty) \setminus \{\alpha(-\infty)\}$, then (by (iii)) there is n such that $\measuredangle_{\alpha(0)}(\phi^n y, \alpha(\infty)) < \varepsilon$. Hence $\alpha(-\infty)$ and $\phi^n y$ are joined by a geodesic $\delta$ which does not bound a flat half plane. Then $\phi^{-n} \circ \delta$ joins $\alpha(-\infty)$ and y and does not bound a flat half plane.

(iv) $\Rightarrow$ (i). Suppose $\alpha$ does bound a flat half plane. Again let $\beta$ be the geodesic in the flat half plane with $\beta(0) = \alpha(0)$ and $\dot\beta(0) \perp \dot\alpha(0)$. It is clear from Euclidean geometry that $d(\alpha(t), \beta(t)) = \sqrt{2}t$ for any $t > 0$.




Now suppose that there is a geodesic $\gamma$ with $\gamma(-\infty) = \beta(\infty)$ and $\gamma(\infty) = \alpha(\infty)$. It is clear from (6.4) that for every $t > 0$ we have $d(\gamma(t),\alpha(t)) \le d(\gamma(0),\alpha(0))$ and $d(\gamma(-t),\beta(t)) \le d(\gamma(0),\beta(0))$. Since

$$\begin{aligned} 2t &= d(\gamma(t),\gamma(-t)) \\ &\le d(\gamma(t),\alpha(t)) + d(\alpha(t),\beta(t)) + d(\beta(t),\gamma(-t)), \end{aligned}$$

we see that for $t > 0$,

$$d(\alpha(t),\beta(t)) \geq 2t - 2d(\gamma(0),\alpha(0)).$$

If t is large enough $2t - 2d(\gamma(0),\alpha(0)) > \sqrt{2}t$. Thus (iv) is impossible if $\alpha$ bounds a flat half plane. □

Finally, a lemma of Ballmann that will be needed in §8 to construct hyperbolic axial isometries.

7.6 LEMMA [5, Lemma 2.11]

Suppose $\alpha$ is a geodesic of H that does not bound a flat half plane and $\phi$ is an isometry of H with no fixed point in H. If $\alpha(-\infty)$ and $\alpha(\infty)$ are fixed points of $\phi$, then $\phi$ is hyperbolic axial with an axis biasymptotic to $\alpha$ or $\bar{\alpha}$.

Proof

Since $\phi$ fixes both $\alpha(-\infty)$ and $\alpha(\infty)$, it is clear that a geodesic $\gamma$ is biasymptotic to $\alpha$ if and only if $\phi\circ\alpha$ is. Consider the set $B = B(\dot{\alpha}(0))$ defined in (6.22). It is clear from (6.23) that we can define a map $\hat{\phi}: B \to B$ as follows. If $p \in B$, then $\hat{\phi}(p)$ is the unique point where the geodesic $\phi\circ\gamma_{p\alpha(\infty)}$, which is biasymptotic to $\alpha$, meets B. Since $\phi$ is an isometry, it is clear from (6.26) that $\hat{\phi}$ is an isometry and in particular continuous.

By (6.23i) and (6.26), B is convex and compact. It is obvious that B is a deformation retract of any closed geodesic ball which contains B.

It follows from the Brouwer fixed point theorem that $\hat{\phi}$ has a fixed point, $p_o$ say, in B.

Let $\beta = \gamma_{p_o \alpha(\infty)}$. Then $\beta$ is biasymptotic to $\alpha$ and so cannot bound a flat half plane (by 6.26). Since $\phi \circ \beta$ is also biasymptotic to $\alpha$, and $p_o$ lies on both $\beta$ and $\phi \circ \beta$, it is clear (see (6.1iii)) that $\phi \circ \beta(t) \equiv \beta(t+\tau)$ for some $\tau$. Since $\phi$ has no fixed point in H, $\tau \neq 0$. If $\tau > 0$, then $\beta$ is an axis of $\phi$. If $\tau > 0$, then $\bar{\beta}$ is an axis of $\phi$. $\square$

## §8. DYNAMICAL PROPERTIES OF THE GEODESIC FLOW

Throughout this section M is a Riemannian manifold with no focal points and H its Riemannian universal cover. We shall see that when M satisfies two conditions, described below, the geodesic flow $\phi_t$ on SM has two properties:

(i) $\phi_t$ is topologically transitive, i.e. there is $v \in SM$ such that $\{\phi_t(v): t > 0\}$ is dense in SM.

(ii) The closed orbits are dense in SM.

These were both proved by Anosov [2] in the case when M is compact and has negative curvature. The conditions that we impose on M can be thought of as weakened forms of compactness and negative curvature respectively.

Firstly, the non-wandering set of $\phi_t$ should be the whole of SM. Obviously (i) and (ii) are impossible without this assumption. It is not unduly restrictive, however; it holds whenever M has finite volume.

Secondly, H should contain a geodesic which does not bound a flat half plane.

The proof that these two conditions on M imply transitivity of the geodesic flow is due to Ballmann [4, 5]; density of the closed orbits was proved by the present author. For a further discussion of these and related results see part D.

The main idea is to study the action of the fundamental group $\pi_1(M)$ on $\bar{H}$. Its relation to the dynamics of the geodesic flow is described in part A. In part B we present the results of Ballmann which enable properties (i) and (ii) to be proved in C.

Both the properties will be needed in §9 when we study ergodicity.

## A. DUALITY AND DYNAMICS

This material is taken from [14, §§2 and 3] and [17, §3].

We want to study the geodesic flow $\phi_t$ on the unit tangent bundle SM. Let us recall some definitions. If $v \in SM$, let $P^+(v)$ be the *positive limit set* of v. Then $w \in P^+(v)$ if and only if there are sequences $v_n \to v$ and $t_n \to \infty$ with $\phi_{t_n}(v_n) \to w$. If $v \in P^+(v)$, then v is *non-wandering*. As usual, let $\Omega$ denote the *non-wandering set* $\{v \in SM: v \in P^+(v)\}$.

We identify the fundamental group $\pi_1(M)$ with the group of covering transformations. This is a properly discontinuous group of isometries of H. In particular, the only covering transformation with a fixed point is the identity. It is clear from §7A that if we extend each of the covering transformations to act on $\bar{H}$, then we obtain a group of homeomorphisms of $\bar{H}$. We also identify $\pi_1(M)$ with this group. Before proceeding, note a point of notation: if $\phi \in \pi_1(M)$, then $\phi_*: TH \to TH$ instead of $T\phi$ will denote its derivative.

The action of $\pi_1(M)$ on $\bar{H}$ can be related to the geodesic flow via the following:

8.1 DEFINITION

Two points $x,y \in H(\infty)$ are *dual* if there is a sequence $\{\phi_n\} \subseteq \pi_1(M)$ such that for every point $p \in H$, $\phi_n^{-1}(p) \to x$ and $\phi_n(p) \to y$. We say that $\{\phi_n\}$ *makes* x and y dual.

Remarks

(i) $x = y$ is allowed.

(ii) We emphasize that p is in H not $H(\infty)$. If M is the flat torus and $H = \mathbb{R}^2$, then $\pi_1(M)$ fixes every point in $H(\infty)$, but it is clear from (8.4) below that antipodal points of $H(\infty)$ must be dual.

(iii) If $\alpha$ is an axis of an axial isometry $\phi \in \pi_1(M)$, we see from (7.1) and the subsequent remarks that $\{\phi^n\}$ makes $\alpha(-\infty)$ and $\alpha(\infty)$ dual.

To verify that $\{\phi_n\}$ makes x and y dual, it is enough to show that $\{\phi_n^{-1}\}$ pushes part of H towards x and $\{\phi_n\}$ pushes part of H towards y.

8.2 LEMMA

Let $x,y \in H(\infty)$. Suppose there are convergent sequences $\{p_n\}$, $\{q_n\} \subseteq H$ and a sequence $\{\phi_n\} \subseteq \pi_1(M)$ such that

$$\phi_n^{-1}(p_n) \to x \text{ and } \phi_n(q_n) \to y.$$

Then $\{\phi_n\}$ makes x and y dual.

Proof

Let $a \in H$. Since $\phi_n$ is an isometry, $\{d(\phi_n^{-1}(p_n),\phi_n^{-1}(a))\}$ and $\{d(\phi_n(q_n),\phi_n(a))\}$ are both bounded. It follows from (6.12) that $\phi_n^{-1}(a) \to x$ and $\phi_n(a) \to y$. □

Now we come to the basic result which relates the dynamics of $\Phi_t$ and the action of $\pi_1(M)$.

8.3 PROPOSITION [15, Proposition 3.7]

Suppose $V$, $W \in SH$ are lifts of $v,w \in SM$. Then $w \in P^+(v)$ if and only if $\gamma_V(\infty)$ and $\gamma_W(-\infty)$ are dual.

In [15] Eberlein assumed that M had non-positive curvature. The results from §6 allow his proof to carry over almost unchanged.

Proof

Let $V \in S_pH$ and $W \in S_qH$.

Suppose firstly that $w \in P^+(v)$. Then there are sequences $v_n \to v$ in SM and $t_n \to \infty$ such that $\Phi_{t_n}(v_n) \to w$. Choose lifts $V_n$ of $v_n$ such that $V_n \to V$. Since $\dot{\gamma}_{V_n}(t_n)$ is a lift of $\Phi_{t_n}(v_n)$, there is a sequence $\{\phi_n\} \subseteq \pi_1(M)$ such that

$$\phi_{n*} \circ \dot{\gamma}_{V_n}(t_n) \to W. \qquad (*)$$

We shall apply (8.2) to show that $\{\phi_n\}$ makes $\gamma_V(\infty)$ and $\gamma_W(-\infty)$ dual. Let

$$p_n = \gamma_{V_n}(0) \text{ and } q_n = \phi_n \circ \gamma_{V_n}(t_n).$$

It is clear that $p_n \to p$ and (*) says that $q_n \to q$. Since $V_n \to V$ and $t_n \to \infty$, it follows from (6.13iv) that

$$\phi_n^{-1}(q_n) = \gamma_{V_n}(t_n) \to \gamma_V(\infty).$$

To see that $\phi_n(p_n) \to \gamma_W(-\infty)$, consider the geodesices $\alpha_n(t) = \phi_n \circ \gamma_{V_n}(t_n - t)$. Then $\alpha_n(t_n) = \phi_n(p_n)$ and (*) says that $\dot{\alpha}_n(0) \to -W$. It follows from (6.13iv) that

$$\phi_n(p_n) = \alpha_n(t_n) \to \gamma_W(-\infty).$$

We now see from (8.2) that $\gamma_V(\infty)$ and $\gamma_W(-\infty)$ are dual.

Conversely, suppose $\{\psi_n\} \subseteq \pi_1(M)$ makes $\gamma_V(\infty)$ and $\gamma_W(-\infty)$ dual. Let $V_n = V(p, \psi_n^{-1}(q))$ and $t_n = d(p, \psi_n^{-1}(q))$. Since $\psi_n^{-1}(q) \to \gamma_V(\infty)$, we see from (6.13i) that $V_n \to V$. Since $\gamma_{V_n} | [0,t_n]$ joins p to $\psi_n^{-1}(q)$, $\psi_n \circ \gamma_{V_n} | [0,t_n]$ joins $\psi_n(p)$ to q. We see that

$$\psi_{n*} \circ \dot{\gamma}_{V_n}(t_n) = -V(q, \psi_n(p)) \to W$$

by (6.13i) since $\psi_n(p) \to \gamma_W(-\infty)$. Let $\{v_n\}$ be the sequence in SM which lifts to $\{V_n\}$. Then $\psi_{n*} \circ \dot{\gamma}_{V_n}(t_n)$ is a lift of $\Phi_{t_n}(v_n)$. We see that $v_n \to v$ and $\Phi_{t_n}(v_n) \to w$. □

8.4 <u>COROLLARY</u>

$\Omega$ = SM if and only if every geodesic $\gamma$ in H has $\gamma(-\infty)$ and $\gamma(\infty)$ dual.

Recall from §1 that $\phi_t$ leaves invariant the Liouville measure $\mu$ which is the measure induced on SM by the Sasaki metric. A standard argument shows that $\Omega$ = SM if $\mu(SM) < \infty$ . Clearly $\mu(SM) < \infty$ if and only if M has finite volume (in the Riemannian measure). In particular, $\Omega$ = SM when M is compact. On the other hand, Eberlein [14, p. 162] has given an example where $\Omega$ = SM but M has infinite volume.

## B. <u>BALLMANN'S CONDITION</u>

8.5 <u>PROPOSITION</u> [5, Theorem 2.13]

Suppose $\Omega$ = SM, and there is a geodesic $\gamma$ in H which does not bound a flat half plane. If U and V are any neighbourhoods in $H(\infty)$ of $\gamma(-\infty)$ and $\gamma(\infty)$ respectively, then there is a hyperbolic axial isometry $\phi \in \pi_1(M)$ with an axis $\alpha$ such that $\alpha(-\infty) \in U$ and $\alpha(\infty) \in V$.

<u>Proof</u>

Let c be greater than the width of the widest flat strip bounded by $\gamma$. Choose $\varepsilon > 0$ small enough so that the properties of (7.4) hold and, if $X_\varepsilon$ and $Y_\varepsilon$ are as in (7.4), then $X_\varepsilon \cap H(\infty) \subseteq U$ and $Y_\varepsilon \cap H(\infty) \subseteq V$. We shall find $\phi \in \pi_1(M) \setminus \{id\}$ with fixed points $x \in X_\varepsilon$ and $y \in Y_\varepsilon$. It will then follow from (7.4) that x and y are joined by a geodesic $\beta$ which does not bound a flat half plane. Since $\phi \neq id$, $\phi$ has no fixed point in H. Hence $x,y \in H(\infty)$ and by (7.6) $\phi$ is a hyperbolic axial isometry with an axis $\alpha$ biasymptotic to $\beta$.

Thus all we need to do is to find $\phi \in \pi_1(M) \setminus \{id\}$ with fixed points in $X_\varepsilon$ and $Y_\varepsilon$. We know from (8.4) that $\gamma(-\infty)$ and $\gamma(\infty)$ are dual. Thus there is a sequence $\{\phi_n\} \subseteq \pi_1(M)$ such that

$$\phi_n^{-1}(\gamma(0)) \to \gamma(-\infty) \text{ and } \phi_n(\gamma(0)) \to \gamma(\infty).$$

We show that for large n, $\phi_n^{-1}(X_\varepsilon) \subseteq X_\varepsilon$ and $\phi_n(Y_\varepsilon) \subseteq Y_\varepsilon$. Since $\phi_n(\gamma(0)) \to \gamma(\infty)$, we see by (6.11) that as $n \to \infty$

$$\measuredangle_{\gamma(0)}(\phi_n(\gamma(0)),\gamma(\infty)) \to 0.$$

Also as $n \to \infty$

$$\measuredangle_{\gamma(0)}(\phi_n(Y_\varepsilon)) = \measuredangle_{\phi_n^{-1}(\gamma(0))}(Y_\varepsilon) \to 0$$

by (7.4iii), since $\phi_n^{-1}(\gamma(0)) \to \gamma(-\infty)$. Similarly it is clear that as $n \to \infty$

$$d(\gamma(0),\phi_n(Y_\varepsilon)) = d(\phi_n^{-1}\cdot\gamma(0),Y_\varepsilon)) \to \infty .$$

Thus for large enough n, $\gamma(0) \notin \phi_n(Y_\varepsilon)$. Since $\gamma(0) \in Y_\varepsilon$, we see that if n is large enough

$$\sphericalangle_{\gamma(0)}(\phi_n(y),\gamma(\infty)) < \sphericalangle_{\gamma(0)}(\phi_n(\gamma(0)),\gamma(\infty)) + \sphericalangle_{\gamma(0)}(\phi_n(Y_\varepsilon))$$

$$< \varepsilon$$

for every $y \in Y_\varepsilon$. We see that for large n, $\phi_n(Y_\varepsilon) \subseteq C(\gamma(0),\gamma(\infty),0,\varepsilon) \subseteq Y_\varepsilon$. A similar argument shows that for all large n, $\phi_n^{-1}(X_\varepsilon) \subseteq X_\varepsilon$.

It is obvious from (6.10) that $X_\varepsilon$ and $Y_\varepsilon$ are homeomorphic to closed n-dimensional discs. It follows from Brouwer's fixed point theorem that $\phi_n$ has fixed points in $X_\varepsilon$ and $Y_\varepsilon$ when n is large. Clearly $\phi_n \neq \mathrm{id}$ if n is large. Thus we can take $\phi = \phi_n$ for some large n. □

We give a name to the property of M that we used in the last proof.

8.6 DEFINITION

M satisfies *Ballmann's condition* if there is a geodesic in the universal cover H which does not bound a flat half plane.

8.7. THEOREM

Suppose M satisfies Ballmann's condition and $\Omega = SM$. Then any points x and y in $H(\infty)$ are dual ($x = y$ is allowed).

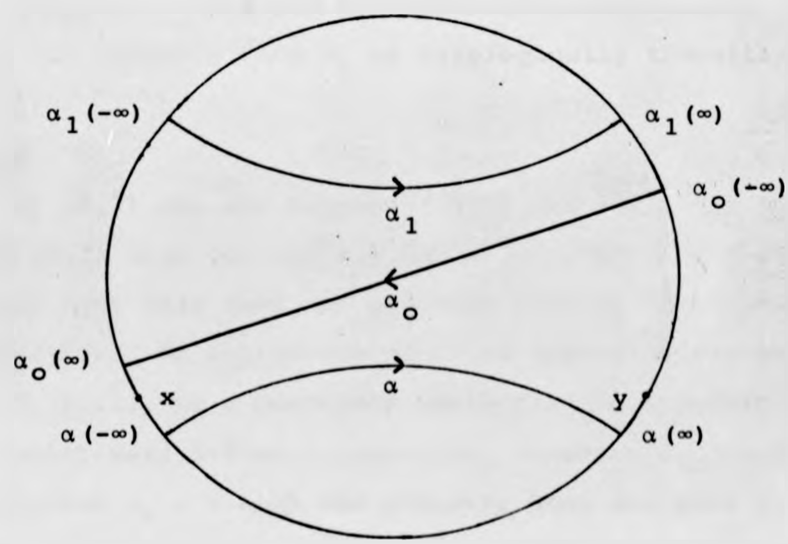
When M has non-positive curvature, this is a special case of Theorem 2.8 of [5].

Proof

By (8.2) it will suffice to prove that for one point $p_o \in H$ there is a sequence $\{\psi_n\} \subseteq \pi_1(M)$ such that

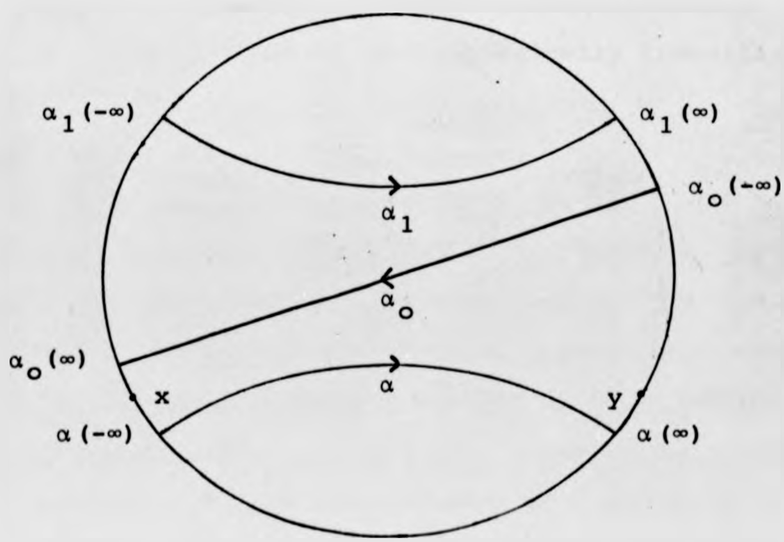
$\psi_n^{-1}(p) \to x$ and $\psi_n(p) \to y$. This will follow if, given any open neighbourhoods in $\bar{H}$, $U_o$ of $x$ and $V_o$ of $y$, we can find $\psi \in \pi_1(M)$ such that $\psi^{-1}(p_o) \in U_o$ and $\psi(p_o) \in V_o$. This in turn will follow if we find an axial isometry $\phi \in \pi_1(M)$ with an axis $\alpha$ such that $\alpha(-\infty) \in U = U_o \cap H(\infty)$ and $\alpha(\infty) \in V = V_o \cap H(\infty)$. For it is clear from (7.1) and the subsequent remarks that $\phi^{-n}(p_o) \to \alpha(-\infty)$ and $\phi^n(p_o) \to \alpha(\infty)$, so we can take $\psi = \phi^n$ for some large enough $n$.

We shall find a hyperbolic axial isometry $\phi \in \pi_1(M)$ with the desired property. By (8.5) there is a hyperbolic axial isometry $\phi_1 \in \pi_1(M)$ with an axis $\alpha_1$. The picture summarizes the argument which will follow.



$\psi_n^{-1}(p) \to x$ and $\psi_n(p) \to y$. This will follow if, given any open neighbourhoods in $\bar{H}$, $U_o$ of $x$ and $V_o$ of $y$, we can find $\psi \in \pi_1(M)$ such that $\psi^{-1}(p_o) \in U_o$ and $\psi(p_o) \in V_o$. This in turn will follow if we find an axial isometry $\phi \in \pi_1(M)$ with an axis $\alpha$ such that $\alpha(-\infty) \in U = U_o \cap H(\infty)$ and $\alpha(\infty) \in V = V_o \cap H(\infty)$. For it is clear from (7.1) and the subsequent remarks that $\phi^{-n}(p_o) \to \alpha(-\infty)$ and $\phi^n(p_o) \to \alpha(\infty)$, so we can take $\psi = \phi^n$ for some large enough $n$.

We shall find a hyperbolic axial isometry $\phi \in \pi_1(M)$ with the desired property. By (8.5) there is a hyperbolic axial isometry $\phi_1 \in \pi_1(M)$ with an axis $\alpha_1$. The picture summarizes the argument which will follow.



By (7.5iii) there is a geodesic $\gamma$ which does not bound a flat half plane and joins x to $\alpha_1(\infty)$. We can choose the direction of $\gamma$ so that $\gamma(-\infty) = \alpha_1(\infty)$ and $\gamma(\infty) = x$. By (8.5) there is a hyperbolic axial isometry $\phi_0 \in \pi_1(M)$ with an axis $\alpha_0$ such that $\alpha_0(-\infty)$ is as close to $\alpha_1(\infty)$ and $\alpha_0(\infty)$ is as close to x as we wish.

By repeating this argument starting from $\alpha_0$ we can find a hyperbolic axial isometry $\phi \in \pi_1(M)$ with an axis $\alpha$ such that $\alpha(-\infty)$ is as close to $\alpha_0(\infty)$ and $\alpha(\infty)$ is as close to y as we wish. Since $\alpha_0(\infty)$ is close to x, we see that we can obtain $\alpha(-\infty) \in U$ and $\alpha(\infty) \in V$. □

## C. APPLICATIONS TO DYNAMICS

8.8 THEOREM [4, Satz 4.6.(3)]

Suppose M satisfies Ballmann's condition and $\Omega = SM$. Then the geodesic flow $\Phi_t$ is topologically transitive on SM.

**Proof**

By (8.7) any two points of $H(\infty)$ are dual. It follows from (8.3) that for any $v \in SM$, $P^+(v) = SM$. It follows easily from this that the geodesic flow is topologically transitive. We follow the proof of [14, Proposition 3.4]. Let $U_1, U_2, \ldots$ be a countable basis for the topology of SM. We inductively define a convergent sequence $\{v_n\} \subseteq SM$ and a sequence $t_n \to \infty$ with the property that for each n,

$$\phi_{t_i}(v_n) \in U_i \quad \text{for} \quad 1 < i < n.$$

We take $v_1 \in U_1$ and $t_1 = 0$. Now suppose that $v_i$ and $t_i$ have already been defined for $1 < i < n$. We will choose $v_{n+1}$ so close to $v_n$ that $\phi_{t_i}(v_{n+1}) \in U_i$ for $1 < i < n$ and $d_{SM}(v_n, v_{n+1}) < \frac{1}{2^n}$, where $d_{SM}$ is distance in the Sasaki metric. Since $P^+(v_n) = SM$ we can, and do, require in addition that $\phi_{t_{n+1}}(v_{n+1}) \in U_{n+1}$ for some $t_{n+1} > t_n + 1$.

Now let $v = \lim_{n\to\infty} v_n$. It is clear that $\phi_{t_n}(v) \in U_n$ for all n. Hence $\{\phi_t(v) : t > 0\}$ is dense in SM. □

Ballmann [4, Satz 7; 5 Theorem 3.5] has proved the stronger result that the geodesic flow is topologically mixing when M is as in (C.8). His argument was for M with non-positive curvature, but it carries over to the case when M has no focal points. We will not present the proof since the result will not be needed in §9.

Instead we turn to the proof of the density of periodic orbits in SM. We know from (8.8) that there is $v \in SM$ for which the forward orbit under the geodesic flow, $\{\dot\gamma_v(t) : t > 0\}$, is dense in SM. Any vector $V \in SH$ which is a lift of such a vector will have the property that

$$\{\phi_* \circ \dot\gamma_V(t) : t > 0,\ \phi \in \pi_1(M)\}$$

is dense in SH. We now show that for such V, $\gamma_V$ cannot bound a flat strip (of positive width).

8.9 <u>LEMMA</u>

Suppose M satisfies Ballmann's condition. Let $\gamma$ be a geodesic in H such that $\{\phi_* \circ \dot{\gamma}(t) : t > 0,\ \phi \in \pi_1(M)\}$ is dense in SH. Then the only geodesics in H biasymptotic to $\gamma$ are translates of $\gamma$.

<u>Proof</u>

We remark that the existence of such a geodesic $\gamma$ implies that $\Omega = SM$.

Let $B = B(\dot{\gamma}(0))$ be the set defined in (6.22). We wish to show that $B = \{\gamma(0)\}$. First we observe that B is compact. If not, we see from (6.26) that $\gamma$ would bound a flat half plane, and it is clear from the density property of $\gamma$ that every geodesic in H would bound a flat half plane.

We now show that for any points p,q in B there is an isometry of the convex set B into itself sending p to q. Since B is compact any isometry of B into itself is surjective, and so has an inverse which is also an isometry [13, p. 314]. So it will suffice to find for each $p \in B$ an isometry $\tau : B \to B$ with $\tau(\gamma(0)) = p$. We shall use:

<u>Sublemma</u>

Suppose $C \subseteq H$ is convex and $\sigma_n : C \to H$ is a sequence of isometries such that $\sigma_n(c_o)$ converges for some $c_o \in C$. Then $\{\sigma_n\}$ has a pointwise convergent subsequence $\{\sigma_{n_k}\}$ and the map $c \to \lim_{k\to\infty} \sigma_{n_k}(c)$ is an isometry of C into H.

<u>Proof</u>

$$\sigma_n = \exp_{\sigma_n(c_o)} \circ d\sigma_n(c_o) \circ \exp_{c_o}^{-1},$$

where $d\sigma_n(c_o)$ is a linear isometry of the subspace of $T_{c_o}H$ spanned by $\exp_{c_o}^{-1}(C)$ into $T_{\sigma_n(c_o)}H$. It is enough to choose $n_k$ so that $\{d\sigma_{n_k}(c_o)\}$ converges. □

It is clear from (6.23) and (6.24) that for any t the map $\tau_t:B \to B(\dot{\gamma}(t))$ defined by translation along the geodesics biasymptotic to $\gamma$ is an isometry. Let w be the unique vector in $S_pH$ such that $\gamma_w$ is biasymptotic to $\gamma$. We can choose sequences $t_n \to \infty$ and $\{\phi_n\} \subseteq \pi_1(M)$ so that $\phi_{n*}\dot{\gamma}(t_n) \to w$. Applying the sublemma to $\{\phi_{n*} \circ \tau_{t_n}\}$ gives an isometry $\tau:B \to H$ with $\tau(\gamma(0)) = p$. We see from (iv) and (iii) of (6.23) that $\tau(B) \subseteq B(w) = B$.

Finally we show that the above is impossible unless B is a single point. Call $p \in B$ extreme if p does not lie in the interior of a geodesic segment contained in B. There is at least one extreme point in B. For, since B is compact, there is $p_o \in B$ as far away from $\gamma(0)$ as possible. The point $p_o$ is extreme since the closed geodesic ball in H with centre $\gamma(0)$ and radius $d(p_o,\gamma(0))$ contains B, is strictly convex by (4.14), and has $p_o$ on its boundary. Clearly any isometry $\tau:B \to B$ maps non-extreme points to non-extreme points. But we know that $\tau$ has an inverse $\tau^{-1}$, which is an isometry as well. So $\tau$ must also map extreme points to extreme points.

It follows that every point in B is extreme. This is impossible if B contains two distinct points. Hence $B = \{\gamma(0)\}$. □

Call a vector in SM *periodic* if its orbit under the geodesic flow is closed, and call a vector in SH periodic if it is a lift of a periodic vector in SM.

8.10 THEOREM

Suppose M satisfies Ballmann's condition and $\Omega = SM$. Then the periodic vectors are dense in SH and hence SM.

Proof

We use the same argument as [15, Theorem 3.10] and [5, Theorem 3.8]. By (8.8) there is a geodesic $\gamma$ in H which satisfies the conditions of (8.9). It will suffice to show that $\dot{\gamma}(0)$ is a limit of periodic vectors in H.

We saw in (8.9) that $\gamma$ does not bound a flat strip. By (8.5) there is a sequence $\{\alpha_n\}$ of axes of hyperbolic axial isometries in $\pi_1(M)$ such that $\alpha_n(-\infty) \to \gamma(-\infty)$ and $\alpha_n(\infty) \to \gamma(\infty)$. It follows from (7.4) that given any $c > 0$, $\alpha_n$ passes within distance c of $\gamma(0)$ for all large enough n. Hence there is a sequence $\{t_n\}$ such that $\alpha_n(t_n) \to \gamma(0)$. Since $\alpha_n(\infty) \to \gamma(\infty)$, it follows from (6.13i) that

$$\dot{\alpha}_n(t_n) = V(\alpha_n(t_n),\alpha_n(\infty)) \to V(\gamma(0),\gamma(\infty)) = \dot{\gamma}(0).$$

But each $\dot{\alpha}_n(t_n)$ is a periodic vector. □

It is clear that the width of the widest flat strip bounded by $\alpha_n$ must approach 0 as $n \to \infty$ ; otherwise $\gamma$ would bound a flat strip of positive width. It follows that for any $\varepsilon > 0$, $\{v \in SH$: $v$ is periodic and $\gamma_v$ does not bound a flat strip of width $\geq \varepsilon\}$ is dense in SH. This generalizes Theorem 3.8 of [5], since we did not assume a priori the existence of a geodesic with no flat strip.

Hurley [32] has proved (8.10) in the case when $M$ is 2-dimensional.

## D. DISCUSSION

The methods used to prove the results in part C are largely inspired by work of Eberlein in the early 1970's. He studied manifolds with the following property.

### 8.11 DEFINITION

A manifold $M$ with no conjugate points satisfies the *axiom of uniform visibility* if the following is true of its universal cover $H$ : given $\varepsilon > 0$ there is $R > 0$ such that if $p$ is a point and $\gamma$ a geodesic in $H$ with $d(p,\gamma) \geq R$ , then the angle subtended by $\gamma$ at $p$ is $\leq \varepsilon$ .

This axiom was introduced by Eberlein and O'Neill [19] as a generalization of negative curvature. If $M$ is compact and has no focal points, uniform visibility holds if and only if $H$ contains no flat totally geodesically embedded plane [14, Theorem 4.1]. Thus for manifolds without focal points Ballmann's condition is a natural weakening of uniform visibility, and it is clear from the introduction to [4] that this is how Ballmann thought of it. There are examples, due to Heintze and Gromov, of compact manifolds with non-positive curvature which satisfy Ballmann's condition but not uniform visibility [5, p. 143]. On the other hand there are manifolds with focal points which satisfy uniform visibility: if $M$ is compact and its geodesic flow is Anosov, then $M$ satisfies uniform visibility [33, p. 11] and $M$ can have focal points [30].

We mention some of Eberlein's results. If $M$ satisfies uniform visibility then it is possible to construct the boundary sphere for the universal cover $H$ (without assuming that $M$ has no focal points) [14, §1]. The geodesic flow is topologically transitive if $\Omega = SM$ [14, Theorem 3.7]. If in addition no geodesic of $H$ bounds a flat strip, then the periodic orbits are dense in $\Omega$ [15, Theorem 3.10 and p. 509].

Eberlein and Ballmann actually consider a wider context than we have here. Instead of studying the fundamental group of a smooth manfold $M$ acting on the universal cover $H$, they consider a simply connected manifold $H$ acted on by a subgroup of its isometry group. The condition that $\Omega = SM$ becomes the duality condition of [10, 11].

Finally, it is natural to ask whether Ballmann's condition is necessary for the geodesic flow to be transitive or for the closed orbits to be dense. If $M$ is compact, it seems very likely that Ballmann's condition is necessary for transitivity but that the closed orbits are always dense. This has recently been proved when $M$ has non-positive curvature; see the remarks in the introduction. The situation seems less clear when $M$ is non-compact.

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## §9. ERGODICITY OF THE GEODESIC FLOW

Throughout this section $M$ will be a compact manifold with no focal points. We will prove the following theorem (9.11 below):

Suppose that $M$ satisfies Ballmann's condition. Then the geodesic flow $\Phi_t$ on $SM$ is ergodic with respect to the Liouville measure $\mu$.

The proof uses two results. The first is a modification of a theorem of Pesin. The second is the fact that the set $\Lambda$, where all of the characteristic exponents of the geodesic flow (except in the flow direction) are non-zero, has positive measure. This was proved by the present author and independently by Ballmann and Brin [6] (in the case when $M$ has non-positive curvature). The main idea of our proof is to show that $\Lambda$ must contain a periodic vector. These two results are in parts B and C. Part A contains results about characteristic exponents.

Notation: $H$ will be the universal cover of $M$; $d_{SM}$ and $d_{SH}$ are distances in the Sasaki metric; $\mu$ is the Liouville measure on $SM$ or $SH$ (defined in §1E). If $W$ is a submanifold immersed in $SM$ or $SH$, $\mu_W$ will denote the measure defined by the Riemannian metric induced on $W$ by the Sasaki metric. In statements involving measures, the measure is $\mu$, unless otherwise specified.

## A. CHARACTERISTIC EXPONENTS

Given $v \in SM$ and a non-zero $\xi \in T_v SM$, we can define the *characteristic exponent* of $\xi$ at $v$,

$$\chi^+(v,\xi) = \limsup_{t \to \infty} \frac{1}{t} \log \|T\Phi_t \xi\| .$$

We want to express this definition in terms of Jacobi fields. The following lemma must be well known, and has been proved in a special case by Pesin [40, p. 801].

9.1 LEMMA

For any non-zero $\xi \in T_v SM$

$$\chi^+(v,\xi) = \limsup_{t \to \infty} \frac{1}{t} \log \|Y_\xi(t)\| ,$$

where $Y_\xi(t) = T\pi \circ T\Phi_t(\xi)$ is the Jacobi field defined in (2.3).

Proof

It is clear from the definition of the Sasaki metric that $\|Y_\xi(t)\| \leq \|T\Phi_t(\xi)\|$, and so

$$\limsup_{t \to \infty} \frac{1}{t} \|Y_\xi(t)\| \leq \chi^+(v,\xi) .$$

To prove the reverse inequality, it will suffice to show that each $\xi \in T_v SM$ has the following property: there is a constant $a(\xi) > 0$ such that

$$\|Y'_\xi(t)\| \leq a(\xi)\|Y_\xi(t)\|$$

for all large enough $t$. Write $\xi = \xi_1 + \xi_2$, where

$\xi_1 \in T''SM$ and $\xi_2 \in T^{\perp}SM$ . Since $Y'_{\xi_1}(t) \equiv 0$ , the property for $\xi$ will follow if we prove it for $\xi_2$ .

We can write $\xi_2 = \eta + \zeta$ , where $\eta , \zeta \in T_v^{\perp}SM$ , $Y_\eta$ is a stable Jacobi field, $Y_\eta(0) = Y_\xi(0)$ and $Y_\zeta(0) = 0$. We show that both $\eta$ and $\zeta$ have the property. Suppose $-k^2$ is a lower bound for the curvature of $M$ . We know that $Y_\zeta$ belongs to the Jacobi map $A(v,t)$ along $\gamma_v$ . If $t > 0$ , $A(v,t)$ is non-singular since there are no conjugate points, and

$$Y'_\zeta(t) = A'A^{-1}(v,t)Y_\zeta(t) \tag{1}$$

Furthermore $A(v,t)$ is non-singular for $0 < t < 2t$ , and so it follows from (3.13) that

$$k \coth(-kt) \leqslant A'A^{-1}(v,t) \leqslant k \coth kt .$$

Hence

$$\|A'A^{-1}(v,t)\| \leqslant k \coth kt . \tag{2}$$

We see from (1) and (2) that if $t$ is large enough,

$$\|Y'_\zeta(t)\| \leqslant 2k\|Y_\zeta(t)\| .$$

Since $Y$ belongs to the stable Jacobi map $D^s(v,t)$ , which is always non-singular by (5.6), the same argument shows that $\eta$ has the property.

Since $Y'_\eta(t)$ and $Y'_\zeta(t)$ need not be orthogonal, it is not immediately clear that the property follows for $\xi_2$ . But, unless $\zeta = 0$ , $\|Y_\zeta(t)\| \to \infty$ as $t \to \infty$ by (5.11), while $\|Y_\eta(t)\|$ is bounded as $t \to \infty$ by (5.8iii).

Using this it is easy to see that when $\zeta \neq 0$ we can take $a(\zeta_2) = a(\zeta) + \varepsilon$ for any $\varepsilon > 0$ ; and when $\zeta = 0$ we can take $a(\xi_2) = a(\eta)$ . □

9.2 <u>COROLLARY</u>

$\chi^+(v,\xi) \leq 0$ if $Y_\xi$ is a stable Jacobi field, and $\chi^+(v,\xi) \geq 0$ if $Y_\xi$ is an unstable Jacobi field.

<u>Proof</u>

By (5.8ii), $||Y_\xi(t)||$ is non-increasing if $Y_\xi$ is stable and non-decreasing if $Y_\xi$ is unstable. □

One is interested in those $v \in SM$ at which $\phi_t$ exhibits hyperbolic behaviour. Define
$\Lambda = \{v \in SM : \chi^+(v,\xi) \neq 0$ for all non-zero $\xi \in T_v^{\perp}SM\}$.

9.3 <u>REMARK</u>

We see that $v \in \Lambda$ if and only if both the inequalities in (9.2) are always strict. Hence if $v \in \Lambda$ , there are n-1 negative characteristic exponents at $v$ . Note also that $v \notin \Lambda$ if $\gamma_v$ bounds a flat strip, for then $\gamma_v$ has a parallel stable Jacobi field.

If $v \in SM$ , define

$$W^s(v) = \{v \in SM : w = v \text{ or } \limsup_{t \to \infty} \frac{1}{t} \log d(\phi_t v, \phi_t w) < 0\}$$

$$W^u(v) = \{v \in SM : w = v \text{ or } \limsup_{t \to \infty} \frac{1}{t} \log d(\phi_{-t} v, \phi_{-t} w) < 0\}$$

$$W^{0s}(v) = \{\Phi_t w : w \in W^s(v) \ , \ t \in \mathbb{R}\} \ ,$$

$$W^{0u}(v) = \{\Phi_t w : w \in W^u(v) \ , \ t \in \mathbb{R}\} \ .$$

The following deep theorem is due to Pesin.

9.4 <u>THEOREM</u> [38]

There is a subset $\tilde{\Lambda} \subseteq \Lambda$ of full measure such that:

(i) If $v \in \tilde{\Lambda}$, then $W^s(v)$ and $W^{0s}(v)$ are smooth injectively immersed submanifolds of dimension $n-1$ and $n$ respectively.

(ii) For almost all $v \in \tilde{\Lambda}$,

$$\mu_{W^s(v)}(W^s(v)\setminus\tilde{\Lambda}) = 0 = \mu_{W^{0s}(v)}(W^{0s}(v)\setminus\tilde{\Lambda}) \ .$$

(iii) The foliations $W^s$ and $W^{0s}$ are absolutely continuous on $\tilde{\Lambda}$.

(iv) Analogous properties hold for $W^u$ and $W^{0u}$.

Characteristic exponents and stable manifolds can also be defined in $SH$, either directly or by lifting from $SM$. We shall use the same notation in both $SM$ and $SH$, except that $\tilde{\Lambda}_{SH}$ will denote the lift to $SH$ of $\tilde{\Lambda}$.

## B. PESIN'S THEOREM

9.5 THEOREM [39, Theorem 9.1]

Suppose that $M$ satisfies the axiom of uniform visibility (see 8.11). Then either $\mu(\Lambda) = 0$ or $\mu(\Lambda) = \mu(SM)$. In the latter case the geodesic flow is ergodic and isomorphic to a Bernoulli flow.

9.6 COROLLARY

The theorem remains true if $M$ satisfies Ballmann's condition.

Proof

This was proved by Ballmann and Brin [6] in the case when $M$ has non-positive curvature. Their proof carries over essentially unchanged when $M$ has no focal points.

Assume $\mu(\Lambda) > 0$. We show first that $\Lambda = SM$ (mod 0). To do this we work in $SH$ and show that $\tilde{\Lambda}_{SH} = SH$ (mod 0). If $v \in SH$, let $L^s(v)$ and $L^u(v)$ be the sets of unit normals to $L^s(v)$ and $L^u(v)$ respectively on the same side as $v$. Lemmas 9.1 - 9.4 of [39] show that there is a subset $\Lambda^* \subseteq \tilde{\Lambda}_{SH}$, of full measure, such that for all $v \in \Lambda^*$, $L^s(v) = W^s(v)$ and $L^u(v) = W^u(v)$; the proofs of these lemmas hold whenever $H$ is the universal cover of a compact manifold with no focal points. Note that if $v \in \Lambda^*$, then

$$W^{0s}(v) = \{w \in SH : \gamma_w(\infty) = \gamma_v(\infty)\} \quad ,$$

and

$$W^{0u}(v) = \{w \in SH : \gamma_w(-\infty) = \gamma_v(-\infty)\} \ .$$

We know from (9.4) that for almost all $w \in \tilde{\Lambda}_{SH}$, almost every vector in $W^{0u}(w)$ (with respect to $\mu_{W^{0u}}$) is in $\tilde{\Lambda}_{SH}$. Call $w \in \tilde{\Lambda}_{SH}$ "good" if it has this property and is in $\Lambda^*$. Almost every $w \in \tilde{\Lambda}$ is "good". Since the foliation $W^s$ is absolutely continuous on $\tilde{\Lambda}_{SH}$, almost every $W^s(v)$ consists almost entirely (with respect to $\mu_{W^s(v)}$) of "good" vectors. In particular there is $v_0 \in \Lambda^*$ with this property. It is clear from the absolute continuity of $W^{0u}$ that (mod 0)

$$\tilde{\Lambda}_{SH} = \bigcup_{w \in W^s(v_0)} W^{0u}(w)$$

$$= \{w' \in SH : \gamma_{w'}(-\infty) = \gamma_w(-\infty), \text{ for some } w \in L^s(v)\} \ .$$

Since it is a lift of $\tilde{\Lambda}$, $\tilde{\Lambda}_{SH}$ is invariant under covering transformations. Hence, if

$$A = \{w' \in SH : \text{for some } \phi \in \pi_1(m),\ \phi \circ \gamma_{w'}(-\infty) \in U\} \ ,$$

where

$$U = \{\gamma_w(-\infty) : w \in L^s(v_0)\} \ ,$$

then

$$\tilde{\Lambda}_{SH} = A \pmod 0 .$$

We show that $A = SH$ . Suppose $w' \in SH$ and $x = \gamma_{w'}(-\infty)$ . By (6.17)

$$U = \{\delta(-\infty) : \delta \text{ is a geodesic in SH with } \delta(\infty) = \gamma_{v_0}(\infty)\} .$$

Since $v_0 \in \tilde{\Lambda}_{SH}$ , $\gamma_{v_0}$ does not bound a flat strip (by 9.3). It follows from (7.4i) that $U$ is a neighbourhood of $\gamma_{v_0}(-\infty)$ in $H(\infty)$ . It is clear from the proof of (8.7) that there is a hyperbolic axial isometry $\psi \in \pi_1(M)$ with an axis $\alpha$ such that $\alpha(\infty) \neq x$ and $\alpha(-\infty) \in U$ . It follows from (7.5iii) that if $n$ is large enough $\psi^{-n}(x) \in U$ . Hence $w' \in A$ and so $A = SH$.

Thus $\Lambda = SM \pmod 0$. Since $\Phi_t$ is topologically transitive on $SM$ by (8.8), it follows from Theorem 9.5 of [38] that $\Phi_t$ is ergodic on $\Lambda$ . It is clear that $W^s$ satisfies the continuity condition in that theorem, since $W^s = L^s(v)$ for almost all $v \in SH$ and $L^s$ is a continuous foliation by (6.20).

Thus $\Phi_t$ is ergodic. The proof that it is Bernoulli is the same as in [39]. □

## C. $\Lambda$ HAS POSITIVE MEASURE

Define $\Lambda_0 = \{v \in SM : \limsup_{t\to\infty} \frac{1}{t} \log \|D^s(v,t)\| < 0$ and $\limsup_{t\to\infty} \frac{1}{t} \log((D^u(v,t))) > 0\}$. It is clear from (9.1) that $\Lambda_0 \subseteq \Lambda$.

### 9.7 LEMMA

If $\Lambda_0$ is non-empty, then $\mu(\Lambda_0) > 0$.

**Proof**

We use an argument of Pesin [40, p. 803]. Let $v_0 \in \Lambda_0$. We can choose $T > 0$ and $\lambda_0$ with $0 < \lambda_0 < 1$ such that

$$\|D^s(v_0,t)\| < \lambda_0 \quad \text{and} \quad ((D^u(v_0,T))) \geq \frac{1}{\lambda_0} .$$

Since $D^s(v,T)$ and $D^u(v,T)$ vary continuously as $v$ varies in $SM$ (by 5.9), we can find an open neighbourhood $U$ of $v_0$ and $\lambda$ with $\lambda_0 \leq \lambda < 1$ such that

$$\|D^s(v,T)\| < \lambda \quad \text{and} \quad ((D^u(v,T))) > \frac{1}{\lambda}$$

for every $v \in U$. Recall that the Liouville measure is invariant under the geodesic flow. By the ergodic theorem there is a set $G \subseteq SM$ with $\mu(G) > 0$ such that for any $v \in G$

$$\liminf_{t\to\infty} \frac{1}{t} \int_0^t \chi_U(\Phi_s(v))\, ds > \frac{1}{2}\mu(U) > 0 ,$$

where $\chi_U$ is the indicator funciton of the set $U$.

Suppose $v \in G$. For $t > 0$ consider sequences

$0 \leq t_1 < t_2 < \dots < t_{n+1} \leq t$ such that $\Phi_{t_i}(v) \in G$ and $t_{i+1} - t_i \geq T$ for $1 \leq i \leq n$. Let $N(t)$ be the largest possible $n$. Clearly,

$$N(t) \geq \frac{1}{2T}\mu(U)t$$

for any large enough $t$. Since $\|D^s(v,t)\| \leq \lambda^{N(t)}$ and $((D^u(v,t))) \geq \lambda^{-N(t)}$, we see that

$$\limsup_{t \to \infty} \frac{1}{t} \log \|D^s(v,t)\| \leq \frac{1}{2T}\mu(U)\log\lambda < 0$$

and

$$\liminf_{t \to \infty} \frac{1}{t} \log((D^u(v,t))) \geq \frac{-1}{2T}\mu(U)\log\lambda > 0 .$$

Thus $G \subseteq \Lambda_0$ and so $\mu(\Lambda_0) > 0$. $\square$

There is a simple criterion to decide whether or not a periodic vector is in $\Lambda_0$.

9.8 LEMMA

Suppose $v \in SM$ is periodic. Then either $v \in \Lambda_0$ or there is a non-zero orthogonal Jacobi field along $\gamma_v$ which is parallel.

Proof

Let $T$ be the period of $\gamma_v$. Then $D^s(v,T): v^\perp \to v^\perp$. We see using (5.8iv) and (5.5iii) that

$$\limsup_{t \to \infty} \frac{1}{t} \log \|D^s(v,t)\| = \limsup_{n \to \infty} \frac{1}{nT} \log \|D^s(v,nT)\|$$

$$= \limsup_{n \to \infty} \frac{1}{nT} \log \|D^s(v,T)^n\|$$

$$= \frac{1}{T} \log \lambda ,$$

where $\lambda$ is the modulus of the largest eigenvalue of $D^s(v,T)$. By (5.8iii), $\|D^s(v,T)w\| \leq \|w\|$ for every $w \in v^\perp$. Hence $\lambda \leq 1$. If $\lambda < 1$,

$$\limsup_{t \to \infty} \frac{1}{t} \log \|D^s(v,t)\| < 0 .$$

If $\lambda = 1$, there is an invariant subspace of $v^\perp$ on which all the eigenvalues of $D^s(v,T)$ have modulus $1$. On this subspace $D^s(v,T)$ is volume preserving but does not expand the length of any vector. It follows that $D^s(v,T)$ is an isometry on this subspace. Hence there is $w \in v^\perp$ with $\|D^s(v,T)^n\| = 1$ for all integers $n$. It follows from (5.8iii) that the stable Jacobi field $D^s(v,t)w$ has constant length $1$. It is parallel by (5.12iii)

Thus we have shown that either

$$\limsup_{t \to \infty} \frac{1}{t} \log \|D^s(v,t)\| < 0$$

or there is a non-zero parallel orthogonal Jacobi field along $\gamma_v$. A similar argument shows that either

$$\liminf_{t \to \infty} \frac{1}{t} \log((D^u(v,t))) > 0$$

or there is a non-zero parallel orthogonal Jacobi field along $\gamma_v$. The lemma follows immediately from these two statements. $\square$

Now we use the preceding lemma and the density of the periodic vectors to show that $\Lambda_0$ is non-empty if $M$ satisfies Ballmann's condition.

9.9 LEMMA

Assume that $M$ satisfies Ballmann's condition. Then the set $\Lambda_0$ contains a periodic vector.

Proof

Suppose not. Then by (9.8) there is a non-zero parallel Jacobi field along each closed geodesic in $M$. Since the periodic vectors are dense in $SM$ by (8.10), we see that there is a non-zero parallel Jacobi field along each geodesic in $H$. We use this to obtain a contradiction. Let $\gamma$ be a geodesic in $H$ such that $\{\phi_*\dot{\gamma}(t) : t \geq 0$ and $\phi \in \pi_1(M)\}$ is dense in $SH$. We shall find a geodesic biasymptotic to $\gamma$, contrary to (8.9).

If $v \in SH$, let $P(v) = \{w \in v^{\perp} : D^{s'}(v,t) \equiv 0\}$. Using (5.12iii) we see that $P(v) = \ker(D^{u'}(v,0) - D^{s'}(v,0))$. Let $k$ be the smallest value of $\dim P(v)$ for $v \in SH$. From the above we have $k \geq 1$. Since $D^{u'}(v,0)$ and $D^{s'}(v,0)$ both vary continuously with $v$ by (5.9), the set $\{v \in SH : \dim P(v) = k\}$ is open. It is clearly invariant under covering transformations. So, by reparametrizing if necessary, we can assume that $P(\dot{\gamma}(0))$ has dimension $k$.

Write $L = L^s(\dot{\gamma}(0))$. If $p \in L$, let $\hat{p}$ be the unit normal to $L$ on the same side as $\dot{\gamma}(0)$. Since $L$ is

$C^2$-embedded in $H$ (this follows from (6.18)), $\hat{p}$ certainly varies continuously on $L$. Thus there is an open neighbourhood $U$ of $\gamma(0)$ in $L$ such that $P(\hat{p})$ has dimension $k$ for every $p \in U$. Using this and (5.9) we see that $P(\hat{p})$ is a continuous $k$-dimensional distribution on $U$. Let $X$ be a continuous vector field on $U$ such that $X(p) \in P(\hat{p})$ for every $p \in U$ and $X(\gamma(0)) \neq 0$. For some $\varepsilon > 0$ there is a $C^1$-curve $\sigma : (-\varepsilon, \varepsilon) \to U$, with $\sigma(0) = \gamma(0)$, which is an integral curve of $X$ [36, p. 3].

Define $\Sigma : (-\varepsilon, \varepsilon) \times \mathbb{R} \to H$ by $\Sigma(s,t) = \gamma_{\hat{\sigma}(s)}(t)$. Since $L$ is $C^2$, $\Sigma$ is $C^1$. We see from (6.19) that for each fixed $s$, $\frac{\partial \Sigma}{\partial s}(s,\cdot)$ is a stable Jacobi field along $\gamma_{\hat{\sigma}(s)}$, and $\frac{\partial \Sigma}{\partial s}(s,0) = X(\sigma(s)) \in P(\hat{\sigma}(s))$. Hence each of the vector fields $\frac{\partial \Sigma}{\partial s}(s,\cdot)$ is parallel along $\gamma_{\hat{\sigma}(s)}$. It follows that for any $t$

$$d(\gamma(t), \Sigma(s,t)) \leq \int_0^s \left\|\frac{\partial \Sigma}{\partial s}(s,t)\right\| ds$$

$$= \int_0^s \|X(\sigma(s))\| ds$$

which is independent of $t$. Hence the geodesics $\gamma$ and $\gamma_{\hat{\sigma}(s)}$ are biasymptotic for any $s \in (-\varepsilon, \varepsilon)$. Since $X(\gamma(0)) \neq 0$, $\sigma(s) \neq \gamma(0)$ for small $s$, and so we have a contradiction to (8.9). $\square$

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9.10 <u>THEOREM</u>

If $M$ satisfies Ballmann's condition, then $\mu(\Lambda) > 0$ .

<u>Proof</u>

This is immediate from (9.7) and (9.9). □

9.11 <u>THEOREM</u>

If $M$ satisfies Ballmann's condition, then the geodesic flow is ergodic and Bernoulli.

<u>Proof</u>

This is immediate from (9.5) and (9.10). □

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